

SPECTRAL BUNDLES

by

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Introduction

In this work I will construct certain general bundles $\langle \mathfrak{M}, \rho, X \rangle$ and $\langle \mathfrak{B}, \eta, X \rangle$ of Hausdorff locally convex spaces associated to a given Banach bundle $\langle \mathfrak{E}, \pi, X \rangle$. Then I will present conditions ensuring the existence of bounded selections $\mathcal{U} \in \Gamma^{x_\infty}(\rho)$ and $\mathcal{P} \in \Gamma^{x_\infty}(\eta)$ both continuous at a point $x_\infty \in X$, such that $\mathcal{U}(x)$ is a C_0 -semigroup of contractions on \mathfrak{E}_x and $\mathcal{P}(x)$ is a spectral projector of the infinitesimal generator of the semigroup $\mathcal{U}(x)$, for every $x \in X$. In a subsequent paper I shall produce examples of the general results presented here.

Here $\mathfrak{M} \doteq \langle \mathfrak{M}, \rho, X \rangle$ and $\mathfrak{B} \doteq \langle \mathfrak{B}, \eta, X \rangle$ are special kind of bundles of Hausdorff locally convex spaces (bundle of Ω -spaces) while $\mathfrak{E} \doteq \langle \mathfrak{E}, \pi, X \rangle$ is a suitable Banach bundle such that the common base space X is a metrizable space. Moreover for all $x \in X$ the stalk $\mathfrak{M}_x \doteq \rho^{-1}(x)$ is a topological subspace of the space $\mathcal{C}_c(\mathbb{R}^+, \mathcal{L}_{S_x}(\mathfrak{E}_x))$ with the topology of compact convergence, of all continuous maps defined on \mathbb{R}^+ and with values in $\mathcal{L}_{S_x}(\mathfrak{E}_x)$, and the stalk $\mathfrak{B}_x \doteq \eta^{-1}(x)$ is a topological subspace of $\mathcal{L}_{S_x}(\mathfrak{E}_x)$. Here $\mathfrak{E}_x \doteq \pi^{-1}(x)$, while $\mathcal{L}_{S_x}(\mathfrak{E}_x)$, is the space, of all linear bounded maps on \mathfrak{E}_x with the topology of uniform convergence over the subsets of $S_x \subset \text{Bounded}(\mathfrak{E}_x)$ which depends, for all $x \in X$, on the same subspace $\mathcal{E} \subseteq \Gamma(\pi)$. Finally $\rho : \mathfrak{M} \rightarrow X$, $\eta : \mathfrak{B} \rightarrow X$, and $\pi : \mathfrak{E} \rightarrow X$ are the projection maps of the respective bundles and $\Gamma^{x_\infty}(\rho)$ is the class of all bounded selections, i.e. maps belonging to the set $\prod_{x \in X} \mathfrak{M}_x$ continuous at x_∞ with respect to the topology on the bundle space \mathfrak{M} , similarly for $\Gamma^{x_\infty}(\eta)$.

A fundamental remark is that the continuity at x_∞ of \mathcal{U} and \mathcal{P} derives by a sort of continuity at the same point of the selection \mathcal{T} of the graphs of the infinitesimal generators of the semigroups \mathcal{U} , where this sort of continuity has to be understood in the following sense. For every $x \in X$ let $\mathcal{T}(x)$ be the graph of the infinitesimal generator

T_x of the semigroup $\mathcal{U}(x)$, then

$$(0.0.1) \quad \begin{cases} \mathcal{T}(x_\infty) = \{\phi(x_\infty) \mid \phi \in \Phi\} \\ \Phi \subseteq \Gamma^{x_\infty}(\pi_{\mathbf{E}^\oplus}) \\ (\forall x \in X)(\forall \phi \in \Phi)(\phi(x) \in \mathcal{T}(x)), \end{cases}$$

where $\Gamma^{x_\infty}(\pi_{\mathbf{E}^\oplus})$ is the class of all bounded selections of the direct sum of bundles $\mathfrak{V} \oplus \mathfrak{V}$ which are continuous at x_∞ .

Hence for any $v \in \text{Dom}(T_{x_\infty})$ there exists a bounded selection ϕ of $\mathfrak{V} \oplus \mathfrak{V}$ such that

$$(0.0.2) \quad \begin{cases} (v, T_{x_\infty} v) = \lim_{x \rightarrow x_\infty} (\phi_1(x), \phi_2(x)) \\ (\phi_1(x), \phi_2(x)) \in \text{Graph}(T_x), \forall x \in X - \{x_\infty\}, \end{cases}$$

where the limit is with respect to the topology on the bundle space of $\mathfrak{V} \oplus \mathfrak{V}$ ¹.

(Θ, \mathcal{E}) —**structure.**

Relation between the topologies on \mathfrak{M} and \mathfrak{B} and that on \mathfrak{E} .

The main general strategy for obtaining the continuity at x_∞ of \mathcal{U} and \mathcal{P} , it is to correlate the topologies on the bundles spaces involved, among others those on \mathfrak{M} and \mathfrak{B} , with that on the space \mathfrak{E} . Due to this fact it is clear that in this work the construction of the right structures has a prominent role.

It is a well-known fact the relative freedom of choice of the topology on the bundle space of any bundle of Ω —spaces. More exactly the possibility of choosing a linear subspace, which is the entire space if X is compact, of the space of all (global) sections of the bundle, i.e. the space of all everywhere defined bounded continuous selections, see [Gie, Theorem 5.9]. This freedom of choice allows the construction of examples of some of the cited correlations of topologies.

Without entering in the definition of the topology of a bundle of Ω —space, we can appreciate how much important it is to choose the “right” set of all sections (in symbols $\Gamma(\zeta)$) of a general bundle $\langle \Omega, \zeta, X \rangle$ of Ω —space, by the following simple but fundamental result, Corollary 1.2.10. Let $f \in \prod_{x \in X}^b \Omega_x$ be any bounded selection and $x_\infty \in X$ such that there exists a section $\sigma \in \Gamma(\zeta)$ such that $\sigma(x_\infty) = f(x_\infty)$. Then by setting

¹ Later we shall see that the topology on the bundle space of $\mathfrak{V} \oplus \mathfrak{V}$ will be constructed in order to ensure that the limit in (0.0.2) is equivalent to say that $v = \lim_{x \rightarrow x_\infty} \phi_1(x)$ and $T_{x_\infty} v = \lim_{x \rightarrow x_\infty} \phi_2(x)$, both limits with respect to the topology on the bundle space \mathfrak{E} .

$f \in \Gamma^{x_\infty}(\zeta)$ iff f is bounded and continuous at x_∞ we have

$$(0.0.3) \quad f \in \Gamma^{x_\infty}(\zeta) \Leftrightarrow (\forall j \in J) \left(\lim_{z \rightarrow x_\infty} \nu_j^z(f(z) - \sigma(z)) = 0 \right),$$

where J is a set such that $\{\nu_j^z \mid j \in J\}$ is a fundamental set of seminorms of the locally convex space $\Omega_z \doteq \zeta^{-1}(z)$ for all $z \in X$. About the problem of establishing if there are sections intersecting f in x_∞ , we can use an important result of the theory of Banach bundles, stating that any Banach bundle over a locally compact base space is “full”, i.e. for any point of the bundle space there exists a section passing on it. For more general bundle of Ω –space we can use the freedom before mentioned.

The criterium I used for determining the correlations between \mathfrak{M} (resp. \mathfrak{B}) and \mathfrak{E} is that of extending to a general bundle two properties of the topology of the space $\mathcal{C}_c(Y, \mathcal{L}_s(Z))$.

Here Z is a normed space, S is a class of bounded subsets of Z , $\mathcal{L}_s(Z)$ is the space of all linear continuous maps on Z with the pointwise topology, finally $\mathcal{C}_c(Y, \mathcal{L}_s(Z))$ is the space of all continuous maps on Y with values in $\mathcal{L}_s(Z)$ with the topology of uniform convergence over the compact subsets of Y .

In order to simplify the notations we here shall consider Z as a Banach space thus $\mathcal{L}_s(Z) = B_s(Z)$, i.e. the space of all bounded linear operators on Z with the strong operator topology.

Let X be a compact space, Y a topological space

$$\begin{aligned} \mathcal{M} &\doteq \{F \in \mathcal{C}_b(X, \mathcal{C}_c(Y, B_s(Z))) \mid (\forall K \in \text{Comp}(Y)) \\ &\quad (C(F, K) \doteq \sup_{(x,s) \in X \times K} \|F(x)(s)\|_{B(Z)} < \infty)\} \\ \mathbf{M}_x &\doteq \overline{\{F(x) \mid F \in \mathcal{M}\}} \end{aligned}$$

Let denote by $\mathfrak{V} \doteq \langle \mathfrak{E}, \pi, X \rangle$ the trivial bundle with constant stalk Z so $\Gamma(\pi) \simeq \mathcal{C}_b(X, Z)$, set

$$(0.0.4) \quad \begin{cases} \mathcal{A}_x \doteq \{\mu_{(v,x)}^K \mid K \in \text{Comp}(Y), v \in \Gamma(\pi)\}, \\ \mu_{(v,x)}^K : \mathbf{M}_x \ni G \mapsto \sup_{s \in K} \|G(s)v(x)\|, \\ \mathbf{M} \doteq \{\langle \mathbf{M}_x, \mathcal{A}_x \rangle\}_{x \in X}. \end{cases}$$

Then by using Lemma 2.2.7 and the cited [Gie, Theorem 5.9] we can construct a bundle of Ω –space say $\mathfrak{V}(\mathbf{M}, \mathcal{M})$ whose stalk at x is the locally convex space $\langle \mathbf{M}_x, \mathcal{A}_x \rangle$ and whose space of bounded continuous sections $\Gamma(\pi_{\mathbf{M}})$ is such that $\Gamma(\pi_{\mathbf{M}}) \simeq \mathcal{M}$.

Let $f \in \prod_{x \in X} \mathbf{M}_x$ be such that $(\forall K \in \text{Comp}(Y))(\sup_{(x,s) \in X \times K} \|f(x)(s)\|_{B(Z)} < \infty)$ then (see Thm. 2.2.8) (1) \Leftrightarrow (2) \Leftrightarrow (3) with

$$(1) (\forall K \in \text{Comp}(Y))(\forall v \in \Gamma(\pi))$$

$$(\lim_{x \rightarrow x_\infty} \sup_{s \in K} \|f(x)(s)v(x) - f(x_\infty)(s)v(x)\| = 0);$$

$$(2) f \in \Gamma^{x_\infty}(\pi_{\mathbf{M}});$$

$$(3) f : X \rightarrow \mathcal{C}_c(Y, B_s(Z)) \text{ continuous at } x_\infty.$$

Moreover (see Thm. 2.2.8) if Y is locally compact for all $t \in Y$

$$(0.0.5) \quad \Gamma(\pi_{\mathbf{M}})_t \bullet \Gamma(\pi) \subseteq \Gamma(\pi).$$

Therefore we constructed two bundles $\mathfrak{B} \doteq \langle \mathfrak{E}, \pi, X \rangle$ and $\mathfrak{B}(\mathbf{M}, \mathcal{M})$ whose topologies are (I) stalkwise related by $\{\mathcal{A}_x\}_{x \in X}$ in (0.0.4) and for which hold (1) \Leftrightarrow (2) and (II) globally related by (0.0.5). Finally $\Gamma^{x_\infty}(\pi_{\mathbf{M}})$ coincide with the subset of all maps $f : X \rightarrow \mathcal{C}_c(Y, B_s(Z))$ continuous at x_∞ such that $(\forall K \in \text{Comp}(Y))(\sup_{(x,s) \in X \times K} \|f(x)(s)\|_{B(Z)} < \infty)$. The natural generalization of the mentioned property (I) leads to the concept of (Θ, \mathcal{E}) –**structure**, see Definition 2.2.2 and Lemma 2.2.5, while the generalization of the property (II) leads to that of **compatible** (Θ, \mathcal{E}) –structure, see Definition 2.2.2.

A similar and more important global correlation between \mathfrak{M} and \mathfrak{E} , this time for the case in which the topology on each stalk \mathfrak{M}_x is that of the pointwise convergence instead of the compact convergence, is that encoded in (4.3.12) in the definition of “invariant” $(\Theta, \mathcal{E}, \mu)$ –structures, see Definition 4.3.7, for a similar definition see Definition 2.2.2. This conclude the discussion about the relationship between the topologies on \mathfrak{M} and \mathfrak{E} , in particular between those on \mathfrak{B} and \mathfrak{E} ²

Kurtz’s General Approximation Theorem

Briefly I shall recall what here has to be understood as a classical stability problem in order to better explain how to generalize it through the language of bundles. The classical stability problem could be so described: fixed a Banach space Z find a sequence $\{S_n : D_n \subseteq Z \rightarrow Z\}$ of possibly unbounded linear operators in Z and a sequence $\{P_n\} \subset B(Z)$ where P_n is a continuous spectral projector of S_n for $n \in \mathbb{N}$, such that if

(A): there exists an operator $S : D \subset Z \rightarrow Z$ such that $S = \lim_{n \rightarrow \infty} S_n$ with respect to a suitable topology or in any other generalized sense,

² Indeed it is sufficient to take $Y = \{pt\}$ i.e. one point.

(B): such that there exists a spectral projector $P \in B(Z)$ of S such that $P = \lim_{n \rightarrow \infty} P_n$ with respect to the strong operator topology.

Here a spectral projector of an operator S in a Banach space is a continuous projector associated to a closed S -invariant subspace Z_0 such that $\sigma(S \upharpoonright Z_0) \subset \sigma(S)$, where $\sigma(T)$ is the spectrum of the operator T .

In Ch IV [Kat] there are many stability theorems in which the previous limit of operators S_n has to be understood with respect to the metric induced by the “gap” between the corresponding closed graphs.

Moreover there are stability theorems even for operators defined in different spaces, obtained by using the concept of *Transition Operators* introduced by Victor Burenkov, see for example [BL1], [BL2] and [BLL], or the results obtained by Massimo Lanza de Cristoforis and Pier Domenico Lamberti by using functional analytic approaches, see for examples [L1], [L2], [LL].

If one try to generalize the classical stability problem to the case in which Z is replaced by any sequence $\{Z_n\}$ of Banach spaces and S_n is defined in Z_n for all n , then he would face the following difficulty: how to adapt the definition of the gap given by Kato to the case of a sequence of different spaces, more in general in which sense to understand the convergence of operators defined in different spaces.

A first step toward the generalization to the case of different spaces of the classical stability problem is the following result Thomas G. Kurtz, [Kur].

THEOREM 0.0.1 (2.1. of [Kur]). *For each n , let $U_n(t)$ be a strongly continuous contraction semigroup defined on L_n with the infinitesimal operator A_n . Let $A = ex - \lim_{n \rightarrow \infty} A_n$. Then there exists a strongly continuous semigroup $U(t)$ on L such that $\lim_{n \rightarrow \infty} U_n(t)Q_n f = U(t)f$ for all $f \in L$ and $t \in \mathbb{R}^+$ if and only if the domain $D(A)$ is dense and the range $R(\lambda_0 - A)$ of $\lambda_0 - A$ is dense in L for some $\lambda_0 > 0$. If the above conditions hold A is the infinitesimal generator of U and we have*

$$(0.0.6) \quad \lim_{n \rightarrow \infty} \sup_{0 \leq s \leq t} \|U_n(s)Q_n f - Q_n U(s)f\|_n = 0,$$

for every $f \in L$ and $t \in \mathbb{R}^+$.

Here $\langle L, \|\cdot\| \rangle$ is a Banach space, $\{\langle L_n, \|\cdot\|_n \rangle\}_{n \in \mathbb{N}}$ is a sequence of Banach spaces, $\{Q_n \in B(L, L_n)\}_{n \in \mathbb{N}}$ such that $\lim_{n \rightarrow \infty} \|Q_n f\|_n = \|f\|$ for all $f \in L$. Let $f \in L$ and

$\{f_n\}_{n \in \mathbb{N}}$ such that $f_n \in L_n$ for every $n \in \mathbb{N}$, thus he set ³

$$(0.0.7) \quad f = \lim_{n \rightarrow \infty} f_n \Leftrightarrow \lim_{n \rightarrow \infty} \|f_n - Q_n f\|_n = 0.$$

Moreover if $A_n : \text{Dom}(A_n) \subseteq L_n \rightarrow L_n$ he defined

$$(Gr) \quad \begin{cases} \text{Graph}(ex - \lim_{n \rightarrow \infty} A_n) \doteq \{\lim_{n \in \mathbb{N}} s_0(n) \mid s_0 \in \Phi_0\} \\ \Phi_0 \doteq \{(f_n, A_n f_n)_{n \in \mathbb{N}} \in (Z \times Z)^{\mathbb{N}} \mid \\ (\forall n \in \mathbb{N})(f_n \in \text{Dom}(A_n)) \wedge (\exists \lim_{n \in \mathbb{N}} (f_n, A_n f_n))\}, \end{cases}$$

where $(f, g) = \lim_{n \in \mathbb{N}} (f_n, A_n f_n)$ iff $f = \lim_{n \in \mathbb{N}} f_n$ and $g = \lim_{n \in \mathbb{N}} A_n f_n$ and all these limits are those defined in (0.0.7). Whenever $\text{Graph}(ex - \lim_{n \rightarrow \infty} A_n)$ is a graph in L he denoted by $ex - \lim_{n \rightarrow \infty} A_n$ the corresponding operator in L .

The Kurtz's approach, just now described, did not make use of the bundle theory, and, except when imposing stronger assumptions, it cannot be implemented in a bundle of Ω -spaces' approach.

The following consideration results fundamental for understanding the strategy behind this work. There is a strong resemblance of (0.0.3) with (0.0.7). More importantly **if the topology on \mathfrak{M} and that on \mathfrak{E} are related by a (Θ, \mathcal{E}) -structure (for a simple example see (0.0.9)) then there exists a "resemblance" of the selection convergence (0.0.3) with the convergence (0.0.6) of the sequence of semigroups $\{U_n\}_{n \in \mathbb{N}}$ to the semigroup U .** ⁴

³ Notice the strong similarity of (0.0.7) with (0.0.3).

⁴ Indeed if we set assume that there exists for every $n \in \mathbb{N}$ $S_n \in B(L_n, L)$ such that $S_n Q_n = Id$ then (0.0.6) would become

$$(0.0.8) \quad (\forall t \in \mathbb{R}^+)(\forall f \in L) \left(\lim_{n \rightarrow \infty} \sup_{0 \leq s \leq t} \|(U_n(s) - Q_n U(s) S_n) Q_n f\|_n = 0 \right).$$

Moreover let $\langle \mathfrak{M}, \rho, X \rangle$ and $\langle \mathfrak{E}, \pi, X \rangle$ be set as in the beginning and assume that $\{\nu_{(K,v)}^z \mid (K,v) \in \text{Comp}(Y), v \in \mathcal{E}\}$ is a fundamental set of seminorms on \mathfrak{M}_z for every $z \in X$, where $\mathcal{E} \subseteq \Gamma(\pi)$. Finally assume that for all $K \in \text{Comp}(Y)$, $v \in \mathcal{E}$ and for all $z \in X$ and $f^z \in \mathfrak{M}_z$

$$(0.0.9) \quad \nu_{(K,v)}^z(f^z) \doteq \sup_{s \in K} \|f^z(s)v(z)\|_z.$$

Thus (0.0.3) would read: if there exists $\sigma \in \Gamma(\rho)$ such that $\sigma(x_\infty) = F(x_\infty)$ then

$$(0.0.10) \quad F \in \Gamma^{x_\infty}(\rho) \Leftrightarrow (\forall K \in \text{Comp}(Y))(\forall v \in \mathcal{E}) \left(\lim_{z \rightarrow x_\infty} \sup_{s \in K} \|(F(z) - \sigma(z))v(z)\|_z = 0 \right).$$

Therefore by setting X the Alexandroff compactification of \mathbb{N} , $x_\infty = \infty$ and for all $n \in \mathbb{N}$

$$(0.0.11) \quad \begin{cases} \mathfrak{E}_n \doteq L_n, \mathfrak{E}_\infty \doteq L \\ \mathfrak{M}_n \doteq \mathcal{C}_c(\mathbb{R}^+, B_s(L_n)) \\ \mathfrak{M}_\infty \doteq \mathcal{C}_c(\mathbb{R}^+, B_s(L)) \\ \mathcal{E} \doteq \{Qf \mid f \in L\}, \end{cases}$$

I used the word resemble due to the difficulty to build a couple of reasonable Kurtz' bundles, i.e. two bundles of Ω -spaces $\langle \mathfrak{E}, \pi, X \rangle$ and $\langle \mathfrak{M}, \rho, X \rangle$ such that X is the Alexandroff compactification of \mathbb{N} and (0.0.11), (0.0.12) hold. In any case it is possible under strong assumptions, see Setion 5.1.

Although the difficulty of constructing a couple of Kurtz's bundles, *the recognition of the before mentioned two resemblances, were sufficient to push me in investigating the possibility of extending the Kurtz's Theorem 0.0.1 in the general framework of bundles of Ω -spaces, by using (Θ, \mathcal{E}) -structure.*

Direct sum of Bundles of Ω -spaces and (Pre)Graph Sections

It should now be clear that in the way of extending the Kurtz's Theorem I am replacing the sequence of Banach spaces $\{L_n\}_{n \in \mathbb{N} \cup \{\infty\}}$ and $\{\mathcal{C}_c(\mathbb{R}^+, B_s(L_n))\}_{n \in \mathbb{N} \cup \{\infty\}}$, where $L_\infty \doteq L$, with a Banach bundle \mathfrak{E} and the bundle of Ω -spaces \mathfrak{M} respectively, while the Kurtz' convergences (0.0.6) and (0.0.7) with convergences of selections on the bundles spaces \mathfrak{M} and \mathfrak{E} respectively. In this view definition (Gr) has to be replaced by that of Pre-Graph sections, see Definition 2.3.2 (essentially (0.0.1)), while the case in which $Graph(ex - \lim_{n \rightarrow \infty} A_n)$ is a graph in L with that of Graph sections, see Definition 2.3.1. Hence it arises as a natural question which topology has to be selected for the bundle space of $\mathfrak{V} \oplus \mathfrak{V}$.

An essential tool used in the definition of $Graph(ex - \lim_{n \rightarrow \infty} A_n)$ in (Gr) is that of convergence of a sequence $(f_n, A_n f_n)$ in the direct sum of the spaces $L_n \oplus L_n$, given by construction as the convergence of both the sequences in L_n in the meaning of (0.0.7).

It is exactly this factorization property the property which I want to preserve when selecting the "right" topology on the bundle space of $\mathfrak{V} \oplus \mathfrak{V}$.

It is a well-known result the solution of this problem in the special case of Banach bundles. I generalize this result for a finite direct sum of bundles of Ω -spaces, by constructing in Theorem 1.2.22 a family of seminorms on the direct sum of Hausdorff

if there exist conditions under which we can obtain that

$$(0.0.12) \quad \begin{cases} \{Qf \mid f \in L\} \subseteq \Gamma(\pi) \\ \{QVS \mid V \in \mathbf{U}(L)\} \subseteq \Gamma(\rho), \end{cases}$$

where $(Qf)(n) \doteq Q_n f$, $(Qf)(\infty) \doteq f$, while $(QVS)(n) \doteq Q_n V S_n$, $(QVS)(\infty) \doteq V$, for all $n \in \mathbb{N}$ and $\mathbf{U}(L)$, is the class of all C_0 -semigroup on L , then by (0.0.10) and (0.0.8) follows that

$$\mathcal{U} \in \Gamma^\infty(\rho),$$

where $\mathcal{U}(n) \doteq U_n$ and $\mathcal{U}(\infty) \doteq U$.

locally convex spaces which is fundamental for any of the following equivalent topologies: the direct sum top. the lc -direct sum top., the box top. and most importantly the product topology.

The result that the fundamental set is direct along with Lemma 1.2.25 allow to define the direct sum of bundles of Ω -spaces as given in Definition 1.2.23.

Finally the fundamental result that the topology on each stalk is the product topology, (fact encrypted in (1.2.8)) the choice given in (1.2.9) of the subset of section of the direct sum of bundles and the general convergence criterium in (0.0.3), allow to show the claimed factorization property in Corollary 1.2.27. I.e. any selection of the direct sum $\bigoplus_{i=1}^n \mathfrak{E}_i$ of bundles is continuous at a point if and only if is continuous at the same point its projection selection of \mathfrak{E}_i for every $i = 1, \dots, n$.

Semigroup Approximation Theorem.

Roughly speaking $\langle \mathcal{T}, x_\infty, \Phi \rangle \in \Delta_\Theta \langle \mathfrak{V}, \mathfrak{W}, \mathcal{E}, X, \mathbb{R}^+ \rangle$ iff $\mathcal{T}(x)$ is the graph of the infinitesimal generator T_x of a C_0 -semigroup $\mathcal{U}(x)$ on \mathfrak{E}_x , for all $x \in X$, holds (0.0.1) and

$$\mathcal{U} \in \Gamma^{x_\infty}(\rho).$$

Thus, according the discussed way of extending the Kurtz' theorem which I intend to perform in this work, to find an element in the class $\Delta_\Theta \langle \mathfrak{V}, \mathfrak{W}, \mathcal{E}, X, \mathbb{R}^+ \rangle$ means to find an extension of Theorem 0.0.1. In the first main result of this work, **Theorem 3.1.16**, has been constructed an element of the class $\Delta_\Theta \langle \mathfrak{V}, \mathfrak{W}, \mathcal{E}, X, \mathbb{R}^+ \rangle$.

Laplace Duality Properties

There are essentially two strong hypothesis to be satisfied in Theorem 3.1.16. In constructing a model for hypothesis $[ii]$ one obtains **Corollary 3.2.1**. In any case the most important one is hypothesis $[i]$, i.e. the assumption that the (Θ, \mathcal{E}) -structure $\langle \mathfrak{V}, \mathfrak{W}, X, \mathbb{R}^+ \rangle$ has the Laplace duality property, see Definition 3.1.9.

Roughly speaking the full Laplace duality property means that the natural action of $\prod_{x \in X} \mathcal{L}(\mathfrak{E}_x)$ over $\prod_{x \in X} \mathfrak{E}_x$, induces, by restriction, an action over $\Gamma(\pi)$ of the Laplace transform of $\Gamma(\rho)$. More exactly

$$(LD) \quad \begin{cases} (\forall \lambda > 0) (\mathcal{L}(\Gamma(\rho))(\cdot)(\lambda) \bullet \Gamma(\pi) \subseteq \Gamma(\pi)) \\ \mathcal{L}(F)(x)(\lambda) \doteq \int_0^\infty e^{-\lambda s} F(x)(s) ds \doteq \int_{\mathbb{R}^+} F(x)(s) d\mu_\lambda(s). \end{cases}$$

See Def. 2.2.1, Def. 3.1.8 and Def. 3.1.9. The implicit assumption is that for all $x \in X$ and $\lambda > 0$

$$\mathfrak{M}_x \subseteq \mathfrak{L}_1(\mathbb{R}^+, \mathcal{L}_{S_x}(\mathfrak{E}_x), \mu_\lambda),$$

where μ_λ is the Laplace measure associated to λ and $\mathfrak{L}_1(\mathbb{R}^+, \mathcal{L}_{S_x}(\mathfrak{E}_x), \mu_\lambda)$ is the space of all μ_λ -integrable maps with values in the locally convex space $\mathcal{L}_{S_x}(\mathfrak{E}_x)$. Among others informations in Prop. 3.3.2 (an application of a result in [Sil]) there are reasonable conditions ensuring the previous inclusion.

I investigated in Section 3.3 a strategy for constructing classes having the full Laplace duality property. Although I worked in this section in a wide generality, here I shall present the applications of the results in the case of interest for the present introduction.

Firstly we note that by construction

$$\Gamma(\pi) \subset \prod_{x \in X} \mathfrak{E}_x,$$

hence the most natural (duality) action to consider over $\Gamma(\pi)$ is the restriction on it of the “standard”⁵ action of

$$\mathcal{L} \left(\prod_{x \in X} \mathfrak{E}_x \right).$$

Secondly we recall that the Laplace duality property is described in terms of the action restricted over $\Gamma(\pi)$ of a subspace of $\prod_{x \in X} \mathfrak{L}_1(\mathbb{R}^+, \mathcal{L}_{S_x}(\mathfrak{E}_x); \mu_\lambda)$.

Therefore the idea is to construct a suitable **locally convex space** \mathfrak{G} and a linear map Ψ such that

$$(0.0.13) \quad \begin{cases} \mathfrak{G} \subset \mathcal{L} \left(\prod_{x \in X} \mathfrak{E}_x \right) \text{ as linear spaces} \\ \Psi : \mathfrak{L}_1(\mathbb{R}^+, \mathfrak{G}, \mu_\lambda) \rightarrow \prod_{x \in X} \mathfrak{L}_1(\mathbb{R}^+, \mathcal{L}_{S_x}(\mathfrak{E}_x); \mu_\lambda), \end{cases}$$

and most importantly such that the following relation between the two actions holds for all $\overline{F} \in \mathfrak{L}_1(\mathbb{R}^+, \mathfrak{G}, \mu_\lambda)$, $x \in X$, $\lambda > 0$ and $v \in \Gamma(\pi)$

$$(0.0.14) \quad \left\langle \int \Psi(\overline{F})(x)(s) d\mu_\lambda(s), v(x) \right\rangle_x = \left\langle \int \overline{F}(s) d\mu_\lambda(s), v \right\rangle(x),$$

see **Corollary 3.3.29**. Here $\mathfrak{L}_1(\mathbb{R}^+, \mathfrak{G}, \mu_\lambda)$ is the space of all μ_λ -integrable maps on \mathbb{R}^+ and at values in the locally convex space \mathfrak{G} , while for any linear space E we denote by $\langle \cdot, \cdot \rangle : \text{End}(E) \times E \rightarrow E$ the standard duality.

⁵ Standard in the following sense $(B, v) \mapsto B(v)$.

It is exactly by (0.0.14) that we can rewrite (LD) as a duality problem. More exactly if $\exists \mathcal{F} \subset \bigcap_{\lambda > 0} \mathfrak{L}_1(\mathbb{R}^+, \mathfrak{G}, \mu_\lambda)$ such that $\Psi(\mathcal{F}) = \Gamma(\rho)$ then

$$(0.0.15) \quad \text{LD} \Leftrightarrow (\forall \lambda > 0) (\langle \mathcal{A}_\lambda, \Gamma(\pi) \rangle \subseteq \Gamma(\pi)),$$

where for all $\lambda > 0$

$$(0.0.16) \quad \mathcal{A}_\lambda \doteq \left\{ \int \overline{F}(s) d\mu_\lambda(s) \mid \overline{F} \in \mathcal{F} \right\} \subset \mathcal{L} \left(\prod_{x \in X} \mathfrak{E}_x \right).$$

There are two advantage of decoding the problem of finding the full Laplace duality property into the invariance problem (0.0.15). Firstly (0.0.15) is a classical problem of invariance of a subspace of a linear topological space for the standard action of a subspace of the space of all linear continuous operators on it. Secondly in (0.0.15) has involved through the definition of \mathcal{A}_λ the space $\mathfrak{L}_1(\mathbb{R}^+, \mathfrak{G}, \mu_\lambda)$, while in (LD) has involved the much more complicate space $\prod_{x \in X} \bigcap_{\lambda > 0} \mathfrak{L}_1(\mathbb{R}^+, \mathcal{L}_{S_x}(\mathfrak{E}_x); \mu_\lambda)$ indeed $\Gamma(\rho)$ belongs to it.

The crucial idea behind Definition 3.3.12 of the space \mathfrak{G} is the use of the concept of **locally convex final topology**. Indeed the well-known property of this topology allows in Lemma 3.3.25, to ensure that for all $v \in \Gamma(\pi)$ the evaluation map

$$(0.0.17) \quad \mathfrak{G} \ni A \mapsto Av \in \prod_{x \in X} \mathfrak{E}_x \text{ is continuous.}$$

And (0.0.14) is essentially a consequence of (0.0.17). Although we are mainly interested to the equality (0.0.14), there is an important result strictly determined by the locally convex final topology on \mathfrak{G} . Namely Theorem 3.3.23 ensures that holds the second statement in (0.0.13) and that for all $\overline{F} \in \mathfrak{L}_1(\mathbb{R}^+, \mathfrak{G}, \mu_\lambda)$

$$\int_x \text{Pr}(\Psi(\overline{F}))(s) d\mu_\lambda(s) = \text{Pr} \circ \left[\int \overline{F}(s) d\mu_\lambda(s) \right] \circ \iota_x.$$

The class $\Delta \langle \mathfrak{V}, \mathfrak{D}, \Theta, \mathcal{E} \rangle$.

Projection Approximation Theorem.

In this section we shall discuss the main result of this work namely **Theorem 4.3.21**, ensuring the existence of an element $\langle \mathcal{T}, \Phi, x_\infty \rangle$ of the class $\Delta \langle \mathfrak{V}, \mathfrak{D}, \Theta, \mathcal{E} \rangle$ such that T_x is the infinitesimal generator of a C_0 -semigroup of contractions on \mathfrak{E}_x and such that there exists a selection

$$(0.0.18) \quad \boxed{\mathcal{P} \in \Gamma^{x_\infty}(\eta),}$$

satisfying (0.0.19) and such that $\mathcal{P}(x)$ is a spectral projector of T_x for all $x \in X$, where T_x is the operator whose graph is $\mathcal{T}(x)$.

We shall start with the definition of the class $\Delta \langle \mathfrak{V}, \mathfrak{D}, \Theta, \mathcal{E} \rangle$, given in Definition 2.3.4. Roughly speaking given a (Θ, \mathcal{E}) –structure $\langle \mathfrak{V}, \mathfrak{D}, X, \{pt\} \rangle$ and denoted $\mathfrak{D} \doteq \langle \mathfrak{B}, \eta, X \rangle$, we say that $\langle \mathcal{T}, \Phi, x_\infty \rangle \in \Delta \langle \mathfrak{V}, \mathfrak{D}, \Theta, \mathcal{E} \rangle$ iff for all $x \in X$ the set $\mathcal{T}(x)$ is a graph in \mathfrak{E}_x , (0.0.1) holds and there exists a selection $\mathcal{P} \in \Gamma^{x_\infty}(\eta)$ continuous at x_∞ such that $\mathcal{P}(x)$ is a projector on \mathfrak{E}_x and for all $x \in X$

$$(0.0.19) \quad \mathcal{P}(x)T_x \subseteq T_x\mathcal{P}(x),$$

where T_x is the operator in \mathfrak{E}_x whose graph is $\mathcal{T}(x)$.

In others words $\langle \mathcal{T}, \Phi, x_\infty \rangle \in \Delta \langle \mathfrak{V}, \mathfrak{D}, \Theta, \mathcal{E} \rangle$ iff \mathcal{T} is a selection of graphs in \mathfrak{E} continuous at x_∞ in the meaning of (0.0.2) and such that there exists a selection \mathcal{P} of projectors on \mathfrak{E} continuous at x_∞ such that \mathcal{P} commutes with \mathcal{T} in the meaning of (0.0.19).

Notice that (0.0.19) implies for all $x \in X$ that the following

$$T_x\mathcal{P}(x)\text{Dom}(T_x) \subseteq \mathcal{P}(x)\mathfrak{E}_x$$

is satisfied by any spectral projector P_x of the operator T_x , indeed by definition for them we have $T_x P_x \mathfrak{E}_x \subseteq P_x \mathfrak{E}_x$. Viceversa whenever T_x is the infinitesimal generator of a C_0 –semigroup $\mathcal{W}_T(x)$ of contractions on \mathfrak{E}_x , the most important case in this work, it results that (0.0.19) is the property satisfied by all the spectral projectors of the form

$$\mathcal{P}(x) \doteq -\frac{1}{2\pi i} \int_{\Gamma} R(-T_x; \zeta) d\zeta,$$

where $R(-T_x; \zeta)$ is the resolvent map of the operator $-T_x$ and Γ is a suitable closed curve on the complex plane. Hence we can consider the commutation in (0.0.19) as the defining property of what we here consider as the “interesting” bundle \mathcal{P} of spectral projectors associated to \mathcal{T} .

Proof of Theorem 4.3.21. Invariant $(\Theta, \mathcal{E}, \mu)$ – structures. Let us describe the principle steps and new structures required on proving Theorem 4.3.21. The first property involved in showing (0.0.18) is that of an invariant (Θ, \mathcal{E}) – structure, see Definition 2.2.2. The characteristic property of an invariant (Θ, \mathcal{E}) – structure $\langle \mathfrak{V}, \mathfrak{W}, X, Y \rangle$ is the following one

$$(0.0.20) \quad \left\{ F \in \prod_{z \in X}^b \mathfrak{M}_z \mid (\forall t \in Y)(F_t \bullet \mathcal{E}(\Theta) \subseteq \Gamma(\pi)) \right\} = \Gamma(\rho),$$

where $\mathfrak{W} \doteq \langle \langle \mathfrak{M}, \gamma \rangle, \rho, X, \mathfrak{R} \rangle$ and $F_t(x) \doteq F(x)(t)$.

The first reason of introducing the concept of invariant structure is that the global relation (0.0.20) implies the following corresponding local one for any $x_\infty \in X$, see Lemma 4.3.15

$$(0.0.21) \quad \left\{ F \in \prod_{z \in X} \mathfrak{M}_z \mid F \bullet \Gamma^{x_\infty}(\pi) \subseteq \Gamma^{x_\infty}(\pi) \right\} \subseteq \Gamma^{x_\infty}(\rho).$$

Hence if we show that

$$(0.0.22) \quad \mathcal{P} \bullet \Gamma^{x_\infty}(\pi) \subseteq \Gamma^{x_\infty}(\pi)$$

the (0.0.18) follows by (0.0.21) for the special case $Y = \{pt\}$.

The presence of the uniform convergence over compact subsets of Y may drastically restrict the class of invariant (Θ, \mathcal{E}) – structures. In order to deal with this problem we introduced the concept of $(\Theta, \mathcal{E}, \mu)$ – structure $\langle \mathfrak{V}, \mathfrak{Q}, X, \mathbb{R}^+ \rangle$, with $\mathfrak{Q} \doteq \langle \langle \mathfrak{H}, \gamma \rangle, \xi, X, \mathfrak{R} \rangle$, Definition 4.3.7. It is essentially the same definition, given for (Θ, \mathcal{E}) – structures but with the following differences

$$(0.0.23) \quad \begin{cases} \mathfrak{H}_x \subseteq \mathfrak{L}_1(Y, \mathcal{L}_{S_x}(\mathfrak{E}_x), \mu) \\ \mathfrak{H}_x \text{ with the pointwise topology.} \end{cases}$$

Thus on each stalk \mathfrak{H}_x the topology is that of pointwise convergence on Y instead of that of compact convergence.

Thus we have an *invariant* $(\Theta, \mathcal{E}, \mu)$ – structures iff:

$$(0.0.24) \quad \left\{ F \in \prod_{z \in X} \mathfrak{H}_z \mid (\forall t \in Y)(F_t \bullet \mathcal{E}(\Theta) \subseteq \Gamma(\pi)) \right\} = \Gamma(\xi),$$

The second reason to be interested to the the invariant $(\Theta, \mathcal{E}, \mu)$ – structure, resides in the fact that it is a necessary assumption in order to have the following implication, see Corollary 4.3.17 and Remark 4.3.18

$$(0.0.25) \quad \mathcal{W}_T \in \Gamma^{x_\infty}(\xi) \Rightarrow \mathcal{P} \bullet \Gamma^{x_\infty}(\pi) \subseteq \Gamma^{x_\infty}(\pi).$$

But as we know the first main result Theorem 3.1.16 states that $\mathcal{W}_T \in \Gamma^{x_\infty}(\rho)$ for a (Θ, \mathcal{E}) – structure which we know to be not a $(\Theta, \mathcal{E}, \mu)$ – structure.

Hence we need a way of connecting the two types of structures. This is performed by the definition of the $(\Theta, \mathcal{E}, \mu)$ – structure $\langle \mathfrak{V}, \mathfrak{V}(\mathbf{M}^\mu, \Gamma(\rho)), X, Y \rangle$ underlying $\langle \mathfrak{V}, \mathfrak{W}, X, Y \rangle$, see Def, 4.3.9.

In view of the property which we are looking for, namely (0.0.26), we have to maintain the vicinity of the original and the underlying structure. This is performed by using [Gie, Theorem 5.9] for constructing bundles with a given subspace of continuous sections. In this way the space $\Gamma(\rho)$ of continuous sections of \mathfrak{W} will be a subspace of the space $\Gamma(\pi_{\mathbf{M}^\mu})$ of all continuous sections of the underlying bundle $\mathfrak{W}(\mathbf{M}^\mu, \Gamma(\rho))$, (the equality if X is compact).

Thus we have, see Proposition 4.3.19,

$$(0.0.26) \quad \Gamma^{x\infty}(\rho) \subseteq \Gamma^{x\infty}(\pi_{\mathbf{M}^\mu}).$$

Finally by Theorem 3.1.16 we know that $\mathcal{W}_T \in \Gamma^{x\infty}(\rho)$ therefore (0.0.22), and (0.0.18), follows by (0.0.25) and (0.0.26). In addition to invariant $(\Theta, \mathcal{E}, \mu)$ – structures, for obtaining (0.0.25) we need another concept, namely that of a μ –related couple $\langle \mathfrak{V}, \mathfrak{Z} \rangle$, Definition 4.3.5 and a bundle type generalization of the Lebesgue Theorem, see Theorem 4.3.6.

Summary of the main results and structures

The main results of this work are the following ones

- (1) An explicit construction of a direct fundamental set of seminorms of the topological direct sum of a finite family of Hausdorff locally convex spaces, (Theorem 1.2.22);
- (2) Characterization of selections of \mathfrak{W} continuous at a point when $\langle \mathfrak{V}, \mathfrak{W}, X, Y \rangle$ is a (Θ, \mathcal{E}) –structure, (Lemma 2.2.5);
- (3) Construction of a (Θ, \mathcal{E}) –structure $\langle \mathfrak{V}, \mathfrak{W}, X, Y \rangle$ and characterization of a subclass of $\Gamma^{x_\infty}(\rho)$ when \mathfrak{V} is trivial, (Theorem 2.2.8)
- (4) Construction of an element in the class $\Delta_\Theta \langle \mathfrak{V}, \mathfrak{W}, \mathcal{E}, X, \mathbb{R}^+ \rangle$, (**Theorem 3.1.16**, Corollary 3.2.1);
- (5) Conditions in order to satisfy the bounded equicontinuity of which in hypothesis (ii) of Theorem 3.1.16 (Corollary 3.2.1);
- (6) Conditions in order to have (3.1.14) (Proposition 3.3.2);
- (7) The technical Lemma 3.3.25 and Theorem 3.3.23;
- (8) **Theorem 3.3.27** and Corollaries 3.3.29 and 3.3.30;
- (9) \mathcal{K} –Uniform Convergence **Theorem 3.3.35**;
- (10) Consequence of being an $\langle \nu, \eta, E, Z, T \rangle$ invariant set V with respect to \mathcal{F} (Proposition 4.1.4);
- (11) Construction of a class $\Delta_\Theta \langle \mathfrak{V}, \mathfrak{D}, \mathfrak{W}, \mathcal{E}, X, \mathbb{R}^+ \rangle$ by using an $\langle \nu, \eta, \mathfrak{G}, K(\Gamma), \mathbb{R}^+ \rangle$ invariant set V with respect to $\{\overline{F}_T\}$ (Corollary 4.2.6);
- (12) A bundle version of the Lebesgue theorem for a μ –related couple $\langle \mathfrak{V}, \mathfrak{Z} \rangle$ (Theorem 4.3.6);
- (13) Technical Lemmas 4.3.12 and 4.3.15;
- (14) Corollary 4.3.14
- (15) Construction of a selection of spectral projectors continuous at a point given a selection of semigroups continuous at the same point (Corollary 4.3.17 and Remark 4.3.18);

- (16) The Main result of this work is the construction of an element in the class $\Delta \langle \mathfrak{V}, \mathfrak{D}, \Theta, \mathcal{E} \rangle$ (**Theorem 4.3.21**).

The main structures defined in this works are the following ones

- (1) Direct sum of bundles of Ω –spaces (Definition 1.2.23);
- (2) (Invariant) (Θ, \mathcal{E}) –structure $\langle \mathfrak{V}, \mathfrak{W}, X, Y \rangle$, (**Definition 2.2.2**);
- (3) Graph section $\langle \mathcal{T}, x_\infty, \Phi \rangle$, (**Definition 2.3.1**);
- (4) Class $\Delta \langle \mathfrak{V}, \mathfrak{D}, \Theta, \mathcal{E} \rangle$, (**Definition 2.3.4**);
- (5) Class $\Delta_\Theta \langle \mathfrak{V}, \mathfrak{W}, \mathcal{E}, X, \mathbb{R}^+ \rangle$, (**Definition 2.4.1**);
- (6) Class $\Delta_\Theta \langle \mathfrak{V}, \mathfrak{D}, \mathfrak{W}, \mathcal{E}, X, \mathbb{R}^+ \rangle$;
- (7) $\langle \mathfrak{V}, \mathfrak{W}, X, \mathbb{R}^+ \rangle$ with the Laplace duality property, (Definition 3.1.9);
- (8) U–Spaces (**Definition 3.3.6**);
- (9) The locally convex space \mathfrak{G} (**Definition 3.3.12**);
- (10) $\langle \nu, \eta, E, Z, T \rangle$ invariant set V with respect to \mathcal{F} (Definition 4.1.3);
- (11) μ –related couple $\langle \mathfrak{V}, \mathfrak{Z} \rangle$ (Definition 4.3.5);
- (12) (Invariant) $(\Theta, \mathcal{E}, \mu)$ – structure $\langle \mathfrak{V}, \mathfrak{Q}, X, Y \rangle$ (Definition 4.3.7);
- (13) (Θ, \mathcal{E}) – structure $\langle \mathfrak{V}, \mathfrak{V}(\mathbf{M}, \Gamma(\xi)), X, Y \rangle$ underlying a $(\Theta, \mathcal{E}, \mu)$ – structure $\langle \mathfrak{V}, \mathfrak{Q}, X, Y \rangle$ (**Definition 4.3.9**).

CHAPTER 1

Direct Sum

1.1. Notations

Let E be a topological vector space, $\langle \mathcal{L}(E), \tau \rangle$ the linear space of all continuous linear maps on E with the topology τ compatible with the linear structure. Thus $\mathbf{U}(\langle \mathcal{L}(E), \tau \rangle)$ is the class of all continuous semigroup morphisms defined on \mathbb{R}^+ and with values in $\mathcal{L}_\tau(E)$, moreover if $\| \cdot \|$ is any seminorm on $\mathcal{L}(E)$ (not necessarily continuous with respect to τ) we set $\mathbf{U}_{\| \cdot \|}(\langle \mathcal{L}(E), \tau \rangle)$ as the subset of all $U \in \mathbf{U}(\langle \mathcal{L}(E), \tau \rangle)$ such that $\|U(s)\| \leq 1$, for all $s \in \mathbb{R}^+$. Finally we set $\mathbf{U}_{is}(\langle \mathcal{L}(E), \tau \rangle)$ as the subset of all $\mathbf{U}(\langle \mathcal{L}(E), \tau \rangle)$ such that there exists a fundamental set Γ of seminorms on E such that $U(s)$ is an isometry with respect to any element in Γ , for all $s \in \mathbb{R}^+$. We use throughout this work the notations of [Gie] and often when referring to Banach bundles those of [FD]. In particular $\langle \langle \mathfrak{E}, \tau \rangle, p, X, \mathfrak{N} \rangle$ or simply $\langle \mathfrak{E}, p, X \rangle$, whenever τ and \mathfrak{N} are known, is a bundle of Ω -spaces (1.5. of [Gie]), where we denote by τ the topology on \mathfrak{E} while with $\mathfrak{N} \doteq \{\nu_j \mid j \in J\}$ the directed set of seminorms on \mathfrak{E} (1.3. of [Gie]). Thus we set $\mathfrak{N}_x \doteq \{\nu_j^x \mid j \in J\}$ with $\nu_j^x \doteq \nu_j \upharpoonright \mathfrak{E}_x$ for all $x \in X$ and $j \in J$. Moreover for any $U \subseteq X$ we shall call the *Space of Sections of $\langle \langle \mathfrak{E}, \tau \rangle, p, X, \mathfrak{N} \rangle$ on U* the linear space $\Gamma_U(p)$ of all continuous bounded selections of p defined on U , namely ¹

$$\Gamma_U(p) \doteq \mathcal{C}(U, \mathfrak{E}) \bigcap \prod_{x \in U}^b \langle \mathfrak{E}_x, \mathfrak{N}_x \rangle$$

where $\mathfrak{E}_x \doteq p^{-1}(x)$,

$$\prod_{x \in U}^b \langle \mathfrak{E}_x, \mathfrak{N}_x \rangle \doteq \left\{ \sigma \in \prod_{x \in U} \mathfrak{E}_x \mid \sup_{x \in U} \nu_j^x(\sigma(x)) < \infty \right\},$$

¹ Notice the similarity of notation with Def. 1.1.1. In any case it will be always clear which definition has to be considered.

where $\mathcal{C}(U, \mathfrak{E})$ is the linear space of all continuous maps $f : U \rightarrow \mathfrak{E}$. Let $U \subseteq X$ and $x \in U$ set

$$\Gamma_U^x(p) \doteq \left\{ f \in \prod_{x \in U} \langle \mathfrak{E}_x, \mathfrak{N}_x \rangle \mid f \text{ is continuous at } x \right\}.$$

So $\Gamma_U(p) = \bigcap_{x \in U} \Gamma_U^x(p)$. We set $\Gamma(p) \doteq \Gamma_X(p)$ and $\Gamma^x(p) \doteq \Gamma_X^x(p)$ for any $x \in X$. The definition of trivial bundle of Ω -spaces is given in 1.8. of [Gie].

All vector spaces are assumed to be over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, Hlcs is for Hausdorff locally convex spaces. We say that $\mathbf{V} \doteq \{\langle V_x, \mathcal{A}_x \rangle\}_{x \in X}$ is a *nice* family of Hlcs if $\{V_x\}_{x \in X}$ is a family of Hlcs such that $\exists J$ for which $\forall x \in X$ the set $\mathcal{A}_x \doteq \{\mu_j^x\}_{j \in J}$ is a directed² family of seminorms on V_x generating the lct on it. For any family of seminorms Γ on a vector space V we shall define the *directed family of seminorms associated to* Γ the following set $\{\sup F \mid F \in \mathcal{P}_\omega(\Gamma)\}$ with the standard order relation of pointwise order on \mathbb{R}^V .

Given two lcs E and F we denote by $\mathcal{L}(E, F)$ the linear space of all linear and continuous maps on E with values in F , and set $\mathcal{L}(E) \doteq \mathcal{L}(E, E)$, moreover by $\text{Pr}(E) \doteq \{P \in \mathcal{L}(E) \mid P \circ P = P\}$ we denote the class of all projectors on E . Let S be a class of bounded subsets of a lcs E , thus $\mathcal{L}_S(E)$ denotes the lcs whose underlining linear space is $\mathcal{L}(E)$ and whose lct is that of uniform convergence over the subsets in S . When E is a normed space and S is the class of all finite parts of E , then $\mathcal{L}_S(E)$ will be denoted by $B_s(E)$, while $B(E)$ denotes $\mathcal{L}(E)$ with the usual norm topology.

Let X, Y be two topological spaces then $\mathcal{C}(X, Y)$ is the set of all continuous maps on X valued in Y , while $\mathcal{C}_c(X, Y)$ is the topological space of all continuous maps on X valued in Y with the topology of uniform convergence over the compact subsets of Y . If Y is a uniform space then $\mathcal{C}^b(X, Y)$ is the space of all bounded maps in $\mathcal{C}(X, Y)$, while $\mathcal{C}_c^b(X, Y) = \mathcal{C}_c(X, Y) \cap \mathcal{C}^b(X, Y)$. If E is a lcs then $\mathcal{C}_c(X, E)$ is a lcs, while if E is a Hlcs and $\text{Comp}(X)$ is a covering of X , for example if X is a locally compact space, then $\mathcal{C}_c(X, E)$ is a Hlcs.

Let Y be a locally compact space, $\mu \in \text{Radon}(Y)$ and $E \in \text{Hlcs}$, then $\mathfrak{L}_1(Y, E, \mu)$ denotes the linear space of all scalarly essentially μ -integrable maps $f : Y \rightarrow E$ such that its integral belongs to E , see [INT, Ch. 6], while $\text{Meas}(Y, \mathfrak{E}_x, \mu)$ denotes the linear space of all μ -measurable maps $f : Y \rightarrow E$.

If S is any set then $\mathcal{P}_\omega(S)$ denotes the class of all finite subsets of S . Finally “u.s.c.” is for upper semicontinuous. Finally we shall give the following

² I.e. $(\forall j_1, j_2 \in J)(\exists j \in J)(\mu_{j_1}^x, \mu_{j_2}^x \leq \mu_j^x)$ with the standard order relation of pointwise order on \mathbb{R}^V .

DEFINITION 1.1.1. Let $\mathfrak{A} \doteq \langle \langle \mathfrak{B}, \tau \rangle, \xi, X, \mathfrak{N} \rangle$ be a bundle of Ω -spaces, $x \in X$ and Q, S subsets of $\prod_{z \in X} \mathfrak{B}_z$. Thus we set

$$Q_S^x \doteq \{H \in Q \mid (\exists F \in S)(H(x) = F(x))\},$$

and $Q_\diamond^x \doteq Q_{\Gamma(\xi)}^x$, while $\Gamma_S^x(\xi) \doteq (\Gamma^x(\xi))_S^x$ and $\Gamma_\diamond^x(\xi) \doteq (\Gamma^x(\xi))_\diamond^x$.

1.2. Direct Sum of Bundles Ω -spaces

1.2.1. Standard construction of Bundles of Ω -spaces.

DEFINITION 1.2.1 ($FM(3) - FM(4)$ ([Gie] §5) and $FM(3^*) - FM(4)$). Let $\mathbf{V} \doteq \{\langle V_x, \mathcal{A}_x \rangle\}_{x \in X}$ be a nice family of Hlcs with $\mathcal{A}_x \doteq \{\mu_j^x\}_{j \in J}$ for all $x \in X$; we say that \mathcal{G} satisfies $FM(3) - FM(4)$ with respect to \mathbf{V} (see [Gie] §5) if $\mathcal{G} \subseteq \prod_{x \in X}^\infty \langle V_x, \mathcal{A}_x \rangle \doteq \{f \in \prod_{x \in X} V_x \mid (\forall j \in J)(\sup_{x \in X} \mu_j^x(f(x)) < \infty)\}$ and

$FM(3)$: $\{f(x) \mid f \in \mathcal{G}\}$ is dense in V_x for all $x \in X$;

$FM(4)$: $X \ni x \mapsto \mu_j^x(f(x))$ is u.s.c. $\forall j \in J$ and $\forall f \in \mathcal{G}$.

Now we shall introduce a stronger condition namely we say that \mathcal{G} satisfies $FM(3^*) - FM(4)$ with respect to \mathbf{V} if $FM(3^*)$ and $FM(4)$ hold where

$$FM(3^*) \quad (\forall x \in X)(\{f(x) \mid f \in \mathcal{G}\} = V_x).$$

REMARK 1.2.2. Let $\mathbf{V}_k \doteq \{\langle V_x, \mathcal{A}_x^k \rangle\}_{x \in X}$, with $k = 1, 2$, be two nice families of Hlcs such that \mathcal{A}_x^1 and \mathcal{A}_x^2 generate the **same** locally convex topology on V_x . Notice that \mathcal{G} satisfies $FM(3) - FM(4)$ with respect to \mathbf{V}_1 **doesn't imply** that \mathcal{G} satisfies $FM(3) - FM(4)$ with respect to \mathbf{V}_2 . Say $\mathcal{A}^k = \{\mu^{j_k} \mid j_k \in J_k\}$, then a necessary condition for which the fact described happens is the following one $(\exists X_0 \subseteq X)(\exists k \in \{1, 2\})(\exists j_s : X_0 \times J_k \rightarrow J_s)(\forall x \in X_0)(\forall j_k \in J_k)(\exists C > 0)(\forall v \in V_x)$

$$\mu_{j_k}^x(v) \leq C \mu_{j_s(x, j_k)}^x(v).$$

In other words the index for some inequalities relating the seminorms on \mathcal{A}^k with those on \mathcal{A}^s , depends on x .

DEFINITION 1.2.3. Let $\mathbf{V}' \doteq \{\langle V_x, \mathcal{A}'_x \rangle\}_{x \in X}$ be a family of Hlcs where $\mathcal{A}'_x \doteq \{\mu_{j_x}^x\}_{j_x \in J_x}$ is a directed family of seminorms on V_x generating the lct on it, for all $x \in X$. Then we

set

$$\begin{cases} J \doteq \bigcup_{x \in X} \{x\} \times J_x, \\ \mu_{(y, j_y)}^x \doteq \begin{cases} \mu_{j_x}^x, & x = y \\ \mathbf{0}, & x \neq y \end{cases} \quad \forall x, y \in X, \forall j_y \in J_y, \\ \mathcal{A}_x \doteq \{\mu_j^x\}_{j \in J}, \quad \forall x \in X. \end{cases}$$

Moreover by setting

$$\prod_{x \in X}^\infty \langle V_x, \mathcal{A}'_x \rangle \doteq \left\{ f \in \prod_{x \in X} V_x \mid \left(\forall \bar{j} \in \prod_{x \in X} J_x \right) \left(\sup_{x \in X} \mu_{\bar{j}(x)}^x(f(x)) < \infty \right) \right\},$$

we say that $\mathcal{G} \subseteq \prod_{x \in X}^\infty \langle V_x, \mathcal{A}'_x \rangle$ satisfies $FM(3) - FM(4')$ with respect to \mathbf{V}' if $FM(3)$ and $FM(4')$ hold where

$$FM(4') \quad (\forall \bar{j} \in J) (\forall f \in \mathcal{G}) \left(X \ni x \mapsto \mu_{\bar{j}(x)}^x(f(x)) \text{ is u.s.c.} \right).$$

REMARK 1.2.4. Definition 1.2.3 ensures the possibility of associating a nice family of Hlcs to any family of Hlcs. Namely let $\mathbf{V}' \doteq \{\langle V_x, \mathcal{A}'_x \rangle\}_{x \in X}$ be a family of Hlcs where $\mathcal{A}'_x \doteq \{\mu_{j_x}^x\}_{j_x \in J_x}$ is a directed family of seminorms on V_x generating the lct on it, for all $x \in X$. Then $\mathbf{V} \doteq \{\langle V_x, \mathcal{A}_x \rangle\}_{x \in X}$ is a nice family of Hlcs, called the nice family of Hlcs associated to \mathbf{V}' . Indeed \mathcal{A}_x generates the lct on V_x moreover it is trivially directed. Moreover

$$\prod_{x \in X}^\infty \langle V_x, \mathcal{A}'_x \rangle = \prod_{x \in X}^\infty \langle V_x, \mathcal{A}_x \rangle.$$

and \mathcal{G} satisfies $FM(3) - FM(4')$ with respect to \mathbf{V}' if and only if it satisfies $FM(3) - FM(4)$ with respect to \mathbf{V} .

DEFINITION 1.2.5 (5.2 – 5.3 of [Gie]). Let $\mathbf{E} = \{\langle \mathbf{E}_x, \mathfrak{N}_x \rangle\}_{x \in X}$ be a nice family of Hlcs with $\mathfrak{N}_x \doteq \{\nu_j^x \mid j \in J\}$ for all $x \in X$. Moreover let \mathcal{E} satisfy $FM(3) - FM(4)$ with respect to \mathbf{E} . Now we shall apply to \mathbf{E} and \mathcal{E} the general procedure described in 5.2 – 5.3 of [Gie] for constructing bundles of Ω -spaces.

We define

$$\mathfrak{B}(\mathbf{E}, \mathcal{E})$$

to be the Bundle generated by the couple $\langle \mathbf{E}, \mathcal{E} \rangle$, if

- (1) $\mathfrak{B}(\mathbf{E}, \mathcal{E}) = \langle \langle \mathfrak{E}(\mathbf{E}), \tau(\mathbf{E}, \mathcal{E}) \rangle, \pi_{\mathbf{E}}, X, \mathfrak{N} \rangle$;
- (2) $\mathfrak{E}(\mathbf{E}) \doteq \bigcup_{x \in X} \{x\} \times \mathbf{E}_x$, $\pi_{\mathbf{E}} : \mathfrak{E}(\mathbf{E}) \ni (x, v) \mapsto x \in X$.
- (3) $\mathfrak{N} = \{\nu_j \mid j \in J\}$, with $\nu_j : \mathfrak{E} \ni (x, v) \mapsto \nu_j^x(v)$;

(4) $\tau(\mathbf{E}, \mathcal{E})$ is the topology on \mathfrak{E}^3 such that for all $(x, v) \in \mathfrak{E}(\mathbf{E})$

$$\mathcal{I}_{(x,v)}^{\tau(\mathbf{E}, \mathcal{E})} \doteq \mathfrak{F}_{\mathcal{B}(\mathbf{E})((x,v))}^{\mathfrak{E}(\mathbf{E})}$$

is the neighbourhood's filter of (x, v) with respect to it. Here $\mathfrak{F}_{\mathcal{B}(\mathbf{E})((x,v))}^{\mathfrak{E}(\mathbf{E})}$ is the filter on $\mathfrak{E}(\mathbf{E})$ generated by the following filter's base

$$\begin{aligned} \mathcal{B}_{\mathbf{E}}((x, v)) \doteq \{ & T_{\mathbf{E}}(U, \sigma, \varepsilon, j) \mid U \in \text{Open}(X), \sigma \in \mathcal{E}, \varepsilon > 0, j \in J \\ & \mid x \in U, \nu_j^x(v - \sigma(x)) < \varepsilon \}, \end{aligned}$$

where

$$(1.2.1) \quad T_{\mathbf{E}}(U, \sigma, \varepsilon, j) \doteq \{(y, w) \in \mathfrak{E}(\mathbf{E}) \mid y \in U, \nu_j^y(w - \sigma(y)) < \varepsilon\};$$

What is important in this construction is the fact that \mathcal{E} is canonically isomorphic to a linear subspace of $\Gamma(\pi_{\mathbf{E}})$ indeed

REMARK 1.2.6. Let $\mathbf{E} = \{\langle \mathbf{E}_x, \mathfrak{N}_x \rangle\}_{x \in X}$ be a nice family of Hlcs with $\mathfrak{N}_x \doteq \{\nu_j^x \mid j \in J\}$ for all $x \in X$. Moreover let \mathcal{E} satisfy $FM(3) - FM(4)$ with respect to \mathbf{E} , and $\mathfrak{V}(\mathbf{E}, \mathcal{E})$ be the bundle generated by the couple $\langle \mathbf{E}, \mathcal{E} \rangle$. Thus according Prop. 5.8 of [Gie] we have

- (1) $\mathfrak{V}(\mathbf{E}, \mathcal{E})$ is a bundle of Ω -spaces;
- (2) with the notations of Definition 1.2.5 $\mathfrak{V}(\mathbf{E}, \mathcal{E})$ is such that
 - (a) $\langle \mathfrak{E}(\mathbf{E})_x, \tau(\mathbf{E}, \mathcal{E}) \rangle$ as topological vector space is isomorphic to $\langle \mathbf{E}_x, \mathfrak{N}_x \rangle$ for all $x \in X$;
 - (b) \mathcal{E} is canonically isomorphic⁴ to a linear subspace of $\Gamma(\pi_{\mathbf{E}})$ and if X is compact and \mathbf{E} is a function module, see [Gie, 5.1.], then $\mathcal{E} \simeq \Gamma(\pi_{\mathbf{E}})$.

REMARK 1.2.7. Let \mathbf{E} be a nice family of Hlcs and let \mathcal{E} satisfy $FM(3-4)$ with respect to \mathbf{E} . Thus for all $U \in \text{Open}(X)$, $\sigma \in \mathcal{E}$, $\varepsilon > 0$, $j \in J$

$$T_{\mathbf{E}}(U, \sigma, \varepsilon, j) = \bigcup_{y \in U} B_{\mathbf{E}_y, j, \varepsilon}(\sigma(y))$$

where for all $s \in \mathbf{E}_y$

$$B_{\mathbf{E}_y, j, \varepsilon}(s) \doteq \{(y, w) \in \mathfrak{E}(\mathbf{E})_y \mid \nu_j^y(w - s) < \varepsilon\}.$$

In others words $T_{\mathbf{E}}(U, \sigma, \varepsilon, j)$ is the σ -deformed cilinder of radius ε , which justifies the name of ε -tube.

³ By applying 5.3. of [Gie] and Ch.1. of [GT] we know that such as topology there exists.

⁴ I.e. $\sigma \leftrightarrow f$ iff $\sigma(x) = (x, f(x))$

1.2.2. Characterizations of Neighbourhood's filters and Sections of Bundles of Ω -spaces. The following are simple but very useful characterizations of the convergence and of a section in a bundle of Ω -spaces.

PROPOSITION 1.2.8. *Let $\mathfrak{V} = \langle \langle \mathfrak{E}, \tau \rangle, \pi, X, \mathfrak{N} \rangle$ be a bundle of Ω -spaces where $\mathfrak{N} \doteq \{\nu_j \mid j \in J\}$. Moreover let $b \in \mathfrak{E}$ and $\{b_\alpha\}_{\alpha \in D}$ a net in \mathfrak{E} . Then $(1) \Leftarrow (2) \Leftarrow (3) \Leftrightarrow (4)$ where*

- (1) $\lim_{\alpha \in D} b_\alpha = b$;
- (2) $(\exists U \in Op(X) \mid U \ni \pi(b))(\exists \sigma \in \Gamma_U(\pi))(\sigma \circ \pi(b) = b)$ such that $\lim_{\alpha \in D} \pi(b_\alpha) = \pi(b)$ and $(\forall j \in J)(\lim_{\alpha \in D} \nu_j(b_\alpha - \sigma(\pi(b_\alpha))) = 0)$;
- (3) $(\exists U' \in Op(X) \mid U' \ni \pi(b))(\exists \sigma' \in \Gamma_U(\pi) \mid \sigma' \circ \pi(b) = b)$ and $(\forall U \in Op(X) \mid U \ni \pi(b))(\forall \sigma \in \Gamma_U(\pi) \mid \sigma \circ \pi(b) = b)$ we have $\lim_{\alpha \in D} \pi(b_\alpha) = \pi(b)$ and $(\forall j \in J)(\lim_{\alpha \in D} \nu_j(b_\alpha - \sigma(\pi(b_\alpha))) = 0)$;
- (4) $(\exists U' \in Op(X) \mid U' \ni \pi(b))(\exists \sigma' \in \Gamma_U(\pi))(\sigma' \circ \pi(b) = b)$ and $\lim_{\alpha \in D} b_\alpha = b$.

Moreover if \mathfrak{V} is locally full then $(1) \Leftrightarrow (4)$.

PROOF. Of course $(3) \rightarrow (2)$. (2) is equivalent to say that $(\exists U \in Op(X) \mid U \ni \pi(b))(\exists \sigma \in \Gamma_U(\pi))(\sigma \circ \pi(b) = b)$ such that $(\forall V \in Op(X) \mid \pi(b) \in V \subseteq U)(\exists \alpha(V) \in D)(\forall \alpha \geq \alpha(V))(\pi(b_\alpha) \in V)$ and $(\forall j \in J)(\forall \varepsilon > 0)(\exists \alpha(V) \in D)(\forall \alpha \geq \alpha(V, j, \varepsilon))(\nu_j(b_\alpha - \sigma(\pi(b_\alpha))) < \varepsilon)$. Set $\alpha(V, j, \varepsilon) \in D$ such that $\alpha(V, j, \varepsilon) \geq \alpha(V)$, $\alpha(V, j, \varepsilon)$ which there exists D being directed, thus we have $(\forall V \in Op(X) \mid \pi(b) \in V \subseteq U)(\forall j \in J)(\forall \varepsilon > 0)(\exists \alpha(V, j, \varepsilon) \in D)$ such that $(\forall \alpha \geq \alpha(V, j, \varepsilon))(\nu_j(b_\alpha - \sigma(\pi(b_\alpha))) < \varepsilon)$ and $\pi(b_\alpha) \in V$. Thus (1) follows by applying 1.5.VII of [Gie]. Finally by applying 1.5.VII of [Gie] (4) (respectively (1) if \mathfrak{V} is locally full) is equivalent to $(\exists U' \in Op(X) \mid U' \ni \pi(b))(\exists \sigma' \in \Gamma_U(\pi))(\sigma' \circ \pi(b) = b)$ and $(\forall \sigma \in \Gamma_U(\pi) \mid \sigma \circ \pi(b) = b)(\forall j \in J)(\forall \varepsilon > 0)(\forall V \in Op(X) \mid \pi(b) \in V \subseteq U)(\exists \bar{\alpha} \in D)(\forall \alpha \geq \bar{\alpha})$ we have $\pi(b_\alpha) \in V$ and $\nu_j(b_\alpha - \sigma(\pi(b_\alpha))) < \varepsilon$ which is (3). \square

Although the following is a simple consequence of the previous result, we give to it the status of Theorem due to its extraordinary importance and use in the whole this work.

THEOREM 1.2.9. *Let $\mathfrak{V} = \langle \langle \mathfrak{E}, \tau \rangle, \pi, X, \mathfrak{N} \rangle$ be a bundle of Ω -spaces, $W \subseteq X$ and indicate $\mathfrak{N} = \{\nu_j \mid j \in J\}$. Moreover let $f \in \mathfrak{E}^W$, $x_\infty \in W$. Then $(1) \Leftarrow (2) \Leftrightarrow (3) \Leftarrow (4) \Leftrightarrow (5)$ where*

- (1) f is continuous in x_∞ ;

(2) $(\exists U \in Op(X) \mid U \ni x_\infty)(\exists \sigma \in \Gamma_U(\pi))(\sigma(x_\infty) = f(x_\infty))$ such that $\nu_j \circ (f - \sigma \circ \pi \circ f) : W \cap U \rightarrow \mathbb{R}$ and $\pi \circ f : W \rightarrow X$ are continuous in x_∞ for all $j \in J$;

(3) $\pi \circ f : W \rightarrow X$ is continuous in x_∞ and $(\exists U \in Op(X) \mid U \ni x_\infty)(\exists \sigma \in \Gamma_U(\pi))(\sigma(x_\infty) = f(x_\infty))$ such that

$$(\forall j \in J) \left(\lim_{y \rightarrow x_\infty, y \in W \cap U} \nu_j(f(y) - \sigma \circ \pi \circ f(y)) = 0 \right);$$

(4) $(\exists U' \in Op(X) \mid U' \ni x_\infty)(\exists \sigma' \in \Gamma_{U'}(\pi))(\sigma'(x_\infty) = f(x_\infty))$ and $(\forall U \in Op(X) \mid U \ni x_\infty)(\forall \sigma \in \Gamma_U(\pi) \mid \sigma(x_\infty) = f(x_\infty))$ we have $\nu_j \circ (f - \sigma) : W \cap U \rightarrow \mathbb{R}$ and $\pi \circ f : W \rightarrow X$ are continuous in x_∞ for all $j \in J$;

(5) $\pi \circ f : W \rightarrow X$ is continuous in x_∞ and $(\exists U' \in Op(X) \mid U' \ni x_\infty)(\exists \sigma' \in \Gamma_{U'}(\pi))(\sigma'(x_\infty) = f(x_\infty))$ and $(\forall U \in Op(X) \mid U \ni x_\infty)(\forall \sigma \in \Gamma_U(\pi) \mid \sigma(x_\infty) = f(x_\infty))$ we have

$$(\forall j \in J) \left(\lim_{y \rightarrow x_\infty, y \in W \cap U} \nu_j(f(y) - \sigma \circ \pi \circ f(y)) = 0 \right);$$

(6) $(\exists U' \in Op(X) \mid U' \ni x_\infty)(\exists \sigma' \in \Gamma_{U'}(\pi))(\sigma'(x_\infty) = f(x_\infty))$ and f is continuous at x_∞ .

Moreover if \mathfrak{V} is locally full then (1) \Leftrightarrow (6) and if it is full we can choose $U = X$ and $U' = X$.

PROOF. (1) is equivalent to say that for each net $\{x_\alpha\}_{\alpha \in D} \subset W$ such that $\lim_{\alpha \in D} x_\alpha = x_\infty$ in W , we have $\lim_{\alpha \in D} f(x_\alpha) = f(x_\infty)$ in \mathfrak{E} . Similarly (2) is equivalent to say that for each net $\{x_\alpha\}_{\alpha \in D} \subset W$ such that $\lim_{\alpha \in D} x_\alpha = x_\infty$ in W , we have $\lim_{\alpha \in D} \pi \circ f(x_\alpha) = \pi \circ f(x_\infty)$ and $(\forall j \in J)(\lim_{\alpha \in D} \nu_j \circ (f - \sigma \circ \pi \circ f)(x_\alpha) = \nu_j \circ (f - \sigma \circ \pi \circ f)(x_\infty))$. Thus (1) \Leftarrow (2) follows by the corresponding one in Proposition 1.2.8 with the positions $(\forall \alpha \in D)(b_\alpha \doteq f(x_\alpha))$ and $b \doteq f(x_\infty)$. Similarly (1) \Leftarrow (5) follows by (1) \Leftarrow (3) of Proposition 1.2.8. Finally (5) \Rightarrow (6) follows by (5) \Rightarrow (1), while if (6) is true then $\pi \circ f$ is continuous at x_∞ indeed π is continuous, then (5) follows by the implication (4) \Rightarrow (3) of Proposition 1.2.8 with the positions $(\forall \alpha \in D)(b_\alpha \doteq f(x_\alpha))$ and $b \doteq f(x_\infty)$. \square

COROLLARY 1.2.10. Let $\mathfrak{V} = \langle \langle \mathfrak{E}, \tau \rangle, \pi, X, \mathfrak{N} \rangle$ be a bundle of Ω -spaces, $W \subseteq X$ and indicate $\mathfrak{N} = \{\nu_j \mid j \in J\}$. Moreover let $f \in \prod_{x \in W} \mathfrak{E}_x$ and $x_\infty \in W$. Then (1) \Leftarrow (2) \Leftrightarrow (3) \Leftarrow (4) \Leftrightarrow (5) \Leftrightarrow (6) where

(1) f is continuous in x_∞ ;

(2) $(\exists U \in Op(X) \mid U \ni x_\infty)(\exists \sigma \in \Gamma_U(\pi))(\sigma(x_\infty) = f(x_\infty))$ such that $\nu_j \circ (f - \sigma) : W \cap U \rightarrow \mathbb{R}$ is continuous in x_∞ for all $j \in J$;

(3) $(\exists U \in Op(X) \mid U \ni x_\infty)(\exists \sigma \in \Gamma_U(\pi))(\sigma(x_\infty) = f(x_\infty))$ such that

$$(\forall j \in J) \left(\lim_{y \rightarrow x_\infty, y \in W \cap U} \nu_j(f(y) - \sigma(y)) = 0 \right);$$

(4) $(\exists U' \in Op(X) \mid U' \ni x_\infty)(\exists \sigma' \in \Gamma_{U'}(\pi))(\sigma'(x_\infty) = f(x_\infty))$ and $(\forall U \in Op(X) \mid U \ni x_\infty)(\forall \sigma \in \Gamma_U(\pi) \mid \sigma(x_\infty) = f(x_\infty))$ we have that $\nu_j \circ (f - \sigma) : W \cap U \rightarrow \mathbb{R}$ is continuous in x_∞ for all $j \in J$;

(5) $(\exists U' \in Op(X) \mid U' \ni x_\infty)(\exists \sigma' \in \Gamma_{U'}(\pi))(\sigma'(x_\infty) = f(x_\infty))$ and $(\forall U \in Op(X) \mid U \ni x_\infty)(\forall \sigma \in \Gamma_U(\pi) \mid \sigma(x_\infty) = f(x_\infty))$ we have

$$(\forall j \in J) \left(\lim_{y \rightarrow x_\infty, y \in W \cap U} \nu_j(f(y) - \sigma(y)) = 0 \right).$$

(6) $(\exists U' \in Op(X) \mid U' \ni x_\infty)(\exists \sigma' \in \Gamma_{U'}(\pi))(\sigma'(x_\infty) = f(x_\infty))$ and f is continuous at x_∞

If \mathfrak{V} is locally full then (1) \Leftrightarrow (6) and if it is full we can choose $U = X$ and $U' = X$.

PROOF. By Theorem 1.2.9 and $\pi \circ f = Id$. □

PROPOSITION 1.2.11. Let \mathfrak{V} be full and such that there exists a linear space E such that for all $x \in X$ there exists a linear subspace $E_x \subseteq E$ such that $\mathfrak{E}_x = \{x\} \times E_x$, and that⁵

$$\{X \ni x \mapsto (x, v) \in \mathfrak{E}_x \mid v \in \bigcap_{x \in X} E_x\} \subset \Gamma(\pi),$$

If $f_0 \in \prod_{x \in X} E_x$ and

$f \in \prod_{x \in X} \mathfrak{E}_x$ such that $f(x) = (x, f_0(x))$ for all $x \in X$ and $f_0(x_\infty) \in \bigcap_{x \in X} E_x$, then

(1) \Leftrightarrow (2) \Leftrightarrow (3), where

(1) f is continuous at x_∞

(2) $(\exists U \in Op(X) \mid U \ni x_\infty)(\exists \sigma \in \mathcal{C}_b(U, E))(\sigma(x_\infty) = f(x_\infty))$ such that for all $j \in J$

$$\lim_{z \rightarrow x_\infty, z \in W \cap U} \nu_j^z(f(z) - \sigma(z)) = 0;$$

(3) for all $j \in J$

$$\lim_{z \rightarrow x_\infty, z \in W \cap U} \nu_j^z((z, f_0(z)) - (z, f_0(x_\infty))) = 0.$$

⁵ An example is when \mathfrak{V} is the trivial bundle.

PROOF. By Corollary 1.2.10 (1) \Leftrightarrow (2). Let (3) hold then (2) is true by setting $\sigma = \tau_{f(x_\infty)} \upharpoonright U$. Let (2) hold then $\nu_j^z((z, f_0(z)) - (z, f(x_\infty))) \leq \nu_j^z((z, f_0(z)) - \sigma(z)) + \nu_j^z(\sigma(z) - \tau_{f(x_\infty)}(z))$, thus (3) follows by (2) and by Corollary 1.2.10 applied to the continuous map $\tau_{f(x_\infty)} \upharpoonright U$. \square

COROLLARY 1.2.12. Let $\mathfrak{V} \doteq \langle \langle \mathfrak{E}, \tau \rangle, \pi, X, \mathfrak{N} \rangle$ be a bundle of Ω -spaces, $W \subseteq X$ and indicate $\mathfrak{N} = \{\nu_j \mid j \in J\}$. Moreover let $f, g \in \prod_{x \in W} \mathfrak{E}_x$ and $x_\infty \in W$. Then if \mathfrak{V} locally full or ν_j is continuous $\forall j \in J$, then (1) \rightarrow (2) where

- (1) $f(x_\infty) = g(x_\infty)$ and f and g are continuous in x_∞ ;
- (2) $(\exists U \in Op(X) \mid x_\infty \in U)$ such that

$$(\forall j \in J) \left(\lim_{y \rightarrow x_\infty, y \in W \cap U} \nu_j(f(y) - g(y)) = 0 \right).$$

Moreover if \mathfrak{V} is full we can choose $U = X$.

PROOF. The statement is trivial in the case of continuity of all the ν_j . Whereas if \mathfrak{V} is locally full by (1) \rightarrow (5) of Corollary 1.2.10 we have $(\exists U \in Op(X))(\exists \sigma \in \Gamma_U(\pi))(\sigma(x_\infty) = f(x_\infty) = g(x_\infty))$ such that

$$(\forall j \in J) \left(\lim_{y \rightarrow x_\infty, y \in W \cap U} \nu_j(f(y) - \sigma(y)) = \lim_{y \rightarrow x_\infty, y \in W \cap U} \nu_j(g(y) - \sigma(y)) = 0 \right).$$

Therefore

$$\lim_{y \rightarrow x_\infty, y \in W \cap U} \nu_j(f(y) - g(y)) \leq \lim_{y \rightarrow x_\infty, y \in W \cap U} \nu_j(f(y) - \sigma(y)) + \lim_{y \rightarrow x_\infty, y \in W \cap U} \nu_j(g(y) - \sigma(y)) = 0.$$

\square

COROLLARY 1.2.13. Let $\langle \langle \mathfrak{E}, \tau \rangle, \pi, X, \mathfrak{N} \rangle$ be a bundle of Ω -spaces, $W \in Op(X)$ and indicate $\mathfrak{N} = \{\nu_j \mid j \in J\}$. Moreover let $f \in \prod_{x \in W}^b \mathfrak{E}_x$. Then (1) \Leftarrow (2) \Leftrightarrow (3) \Leftarrow (4) \Leftrightarrow (5) where

- (1) $f \in \Gamma_W(\pi)$;

- (2)

$$(\forall x \in W)(\exists U_x \in Op(X) \mid U_x \ni x)(\exists \sigma_x \in \Gamma_{U_x}(\pi))(\sigma_x(x) = f(x))$$

such that $\nu_j \circ (f - \sigma_x)$ is continuous in x , $\forall j \in J$;

- (3)

$$(\forall x \in W)(\exists U_x \in Op(X) \mid U_x \ni x)(\exists \sigma_x \in \Gamma_{U_x}(\pi))(\sigma_x(x) = f(x))$$

such that $(\forall j \in J)(\lim_{y \rightarrow x, y \in W \cap U_x} \nu_j(f(y) - \sigma_x(y)) = 0)$;

(4)

$$(\forall x \in W)(\exists U'_x \in Op(X) \mid U'_x \ni x)(\exists \sigma'_x \in \Gamma_{U'_x}(\pi))(\sigma'_x(x) = f(x))$$

and

$$(\forall U_x \in Op(X) \mid U_x \ni x)(\forall \sigma_x \in \Gamma_{U_x}(\pi) \mid \sigma_x(x) = f(x))$$

we have that $\nu_j \circ (f - \sigma_x)$ is continuous in x for all $x \in W$ and $j \in J$;

(5)

$$(\forall x \in W)(\exists U'_x \in Op(X) \mid U'_x \ni x)(\exists \sigma'_x \in \Gamma_{U'_x}(\pi))(\sigma'_x(x) = f(x))$$

and

$$(\forall x \in W)(\forall U_x \in Op(X) \mid U_x \ni x)(\forall \sigma_x \in \Gamma_{U_x}(\pi) \mid \sigma_x(x) = f(x))$$

we have $(\forall j \in J)(\lim_{y \rightarrow x, y \in W \cap U_x} \nu_j(f(y) - \sigma_x(y)) = 0)$.

PROOF. By Corollary 1.2.10. □

In the case in which the bundle is locally full we can give some useful characterizations of the Neighbourhood's filter \mathcal{I}_α^τ of any point α in the bundle space $\langle \mathfrak{E}, \tau \rangle$.

DEFINITION 1.2.14 (ε -Tubes). *Let $\mathfrak{P} \doteq \langle \langle \mathfrak{E}, \tau \rangle, p, X, \mathfrak{N} \rangle$ be a locally full bundle of Ω -spaces, and let us denote $\mathfrak{N} \doteq \{\nu_j \mid j \in J\}$. Set*

$$\begin{cases} \mathcal{K}^{loc} \doteq \prod_{\alpha \in \mathfrak{E}} \mathcal{K}_\alpha^{loc} \\ \mathcal{K}_\alpha^{loc} \doteq \{(U, \sigma_U) \mid U \in Op(X), \sigma_U \in \Gamma_U(p) \mid p(\alpha) \in U, \sigma_U(p(\alpha)) = \alpha\}. \end{cases}$$

Moreover $\forall \alpha \in \mathfrak{E}$ and $\forall l \in \mathcal{K}^{loc}$ set

$$\begin{cases} \mathcal{B}_l^{loc}(\alpha) \doteq \{T^{loc}(V, \mathfrak{l}_2(\alpha), \varepsilon, j) \mid V \in Op(X), \varepsilon > 0, j \in J \mid p(\alpha) \in V \subseteq \mathfrak{l}_1(\alpha)\}, \\ T^{loc}(U, \sigma_U, \varepsilon, j) \doteq \{\beta \in \mathfrak{E} \mid p(\beta) \in U, \nu_j(\beta - \sigma_U(p(\beta))) < \varepsilon\}, \end{cases}$$

$$(\forall U \in Op(X))(\forall j \in J)(\forall \varepsilon > 0)(\forall \sigma_U \in \Gamma_U(p)).$$

If \mathfrak{P} is a full bundle then we can set

$$\begin{cases} \mathcal{K} \doteq \prod_{\alpha \in \mathfrak{E}} \mathcal{K}_\alpha \\ \mathcal{K}_\alpha \doteq \{(U, \sigma) \mid U \in Op(X), \sigma \in \Gamma(p) \mid p(\alpha) \in U, \sigma(p(\alpha)) = \alpha\}. \end{cases}$$

Moreover $\forall \alpha \in \mathfrak{E}$ and $\forall l \in \mathcal{K}$ set

$$\begin{cases} \mathcal{B}_l(\alpha) \doteq \{T(V, l_2(\alpha), \varepsilon, j) \mid V \in Op(X), \varepsilon > 0, j \in J \mid p(\alpha) \in V \subseteq l_1(\alpha)\}, \\ T(U, \sigma, \varepsilon, j) \doteq T^{loc}(U, \sigma \upharpoonright U, \varepsilon, j), \end{cases}$$

$(\forall U \in Op(X))(\forall j \in J)(\forall \varepsilon > 0)(\forall \sigma \in \Gamma(p))$. Any set $T^{loc}(U, \sigma \upharpoonright U, \varepsilon, j)$ for a fixed $\varepsilon > 0$ is called ε -Tube.

REMARK 1.2.15. Notice that $(\forall U \in Op(X))(\forall j \in J)(\forall \varepsilon > 0)(\forall \sigma_U \in \Gamma_U(p))$

$$T^{loc}(U, \sigma_U, \varepsilon, j) = \bigcup_{y \in U} B_{\mathfrak{E}_y, j, \varepsilon}(\sigma_U(y))$$

where for all $\gamma \in \mathfrak{E}_y$

$$B_{\mathfrak{E}_y, j, \varepsilon}(\gamma) \doteq \{\beta \in \mathfrak{E}_y \mid \nu_j^y(\beta - \gamma) < \varepsilon\}.$$

COROLLARY 1.2.16 (Neighbourhood's filter \mathcal{I}_α^τ). Let $\mathfrak{P} \doteq \langle \langle \mathfrak{E}, \tau \rangle, p, X, \mathfrak{N} \rangle$ be a bundle of Ω -spaces

- (1) if \mathfrak{P} is locally full $\forall \alpha \in \mathfrak{E}$ and $\forall l \in \mathcal{K}^{loc}$ the class $\mathcal{B}_l^{loc}(\alpha)$ is a basis of a filter moreover

$$\mathfrak{F}_{\mathcal{B}_l^{loc}(\alpha)}^\mathfrak{E} = \mathcal{I}_\alpha^\tau;$$

- (2) if \mathfrak{P} is full or locally full over a completely regular space then $\forall \alpha \in \mathfrak{E}$ and $\forall l \in \mathcal{K}$ the class $\mathcal{B}_l(\alpha)$ is a basis of a filter moreover

$$\mathfrak{F}_{\mathcal{B}_l(\alpha)}^\mathfrak{E} = \mathcal{I}_\alpha^\tau.$$

Here \mathcal{I}_α^τ is the neighbourhood's filter of α in the topological space $\langle \mathfrak{E}, \tau \rangle$.

PROOF. Statement (1) follows by applying 1.5. VII of [Gie], while statement (2) follows by statement (1) and the fact that for all $U \in Op(X)$ and $\sigma \in \Gamma(p)$ we have $\sigma \upharpoonright U \in \Gamma_U(p)$ and $T(U, \sigma, \varepsilon, j) \doteq T^{loc}(U, \sigma \upharpoonright U, \varepsilon, j)$, for all $j \in J$ and $\varepsilon > 0$. \square

DEFINITION 1.2.17. Let $\mathbf{E} \doteq \{\langle \mathbf{E}_x, \mathfrak{N}_x \rangle\}_{x \in X}$ be a nice family of Hlcs with $\mathfrak{N}_x \doteq \{\nu_{j \in J}^x\}$ for all $x \in X$. Moreover let \mathcal{E} satisfy $FM(3^*) - FM(4)$ with respect to \mathbf{E} . Set as usual $\mathfrak{E}(\mathbf{E}) \doteq \bigcup_{x \in X} \{x\} \times \mathbf{E}_x$ and

$$\begin{cases} \mathcal{K}^\mathcal{E} \doteq \prod_{(x,v) \in \mathfrak{E}} \mathcal{K}_{(x,v)}^\mathcal{E} \\ \mathcal{K}_{(x,v)}^\mathcal{E} \doteq \{(U, f) \mid U \in Op(X), f \in \mathcal{E} \mid x \in U, f(x) = v\}. \end{cases}$$

Moreover $\forall (x, v) \in \mathfrak{E}(\mathbf{E})$ and $\forall l \in \mathcal{K}^\mathcal{E}$ set

(1.2.2)

$$\mathcal{B}_l^\mathcal{E}((x, v)) = \{T_\mathbf{E}(V, l_2((x, v)), \varepsilon, j) \mid \varepsilon > 0, j \in J, V \in Op(X) \mid x \in V \subseteq l_1((x, v))\}.$$

Here $T_\mathbf{E}(V, l_2((x, v)), \varepsilon, j)$ has been defined in (1.2.1).

COROLLARY 1.2.18 (Neighbourhood's filter $\mathcal{I}_{(x,v)}^{\tau(\mathbf{E}, \mathcal{E})}$). *Assume notations in Definition 1.2.5 and in Definition 1.2.17. Let $\mathbf{E} \doteq \{\langle E_x, \mathfrak{N}_x \rangle\}_{x \in X}$ be a nice family of Hlcs with $\mathfrak{N}_x \doteq \{\nu_{j \in J}^x\}$ for all $x \in X$, moreover let \mathcal{E} satisfy $FM(3^*) - FM(4)$ with respect to \mathbf{E} . Then $\mathfrak{V}(\mathbf{E}, \mathcal{E}) \doteq \langle \langle \mathfrak{E}(\mathbf{E}), \tau(\mathbf{E}, \mathcal{E}) \rangle, \pi_\mathbf{E}, X, \mathfrak{N} \rangle$ is a full bundle of Ω -spaces and $\forall (x, v) \in \mathfrak{E}(\mathbf{E})$*

$$\mathfrak{F}_{\mathcal{B}_l^\mathcal{E}((x,v))}^{\mathfrak{E}(\mathbf{E})} = \mathcal{I}_{(x,v)}^{\tau(\mathbf{E}, \mathcal{E})}.$$

Here $\mathcal{I}_{(x,v)}^{\tau(\mathbf{E}, \mathcal{E})}$ is the neighbourhood's filter of (x, v) in the topological space $\langle \mathfrak{E}(\mathbf{E}), \tau(\mathbf{E}, \mathcal{E}) \rangle$.

PROOF. By Theorem 5.9. of [Gie] \mathcal{E} and $\Gamma(p_1)$ are canonically isomorphic as linear spaces, so $\mathfrak{V}(\mathbf{E}, \mathcal{E})$ is full by $FM(3^*)$. The statement hence follows by statement (2) of Corollary 1.2.16. \square

The following corollaries provide conditions under which the topologies over two bundle spaces are equal.

COROLLARY 1.2.19. *Let $\langle \langle \mathfrak{E}, \tau_k \rangle, p_k, X, \mathfrak{N}_k \rangle$ be a full bundle of Ω -spaces or a locally full bundle over a completely regular space X , for $k = 1, 2$. If there exists a map p such that $p = p_1 = p_2$ (equality as maps) and $\Gamma(p_1) = \Gamma(p_2)$ then $\tau_1 = \tau_2$.*

PROOF. By statement (2) of Corollary 1.2.16. \square

COROLLARY 1.2.20. *Let us assume the hypothesis and notations in Corollary 1.2.18, moreover let $\mathfrak{P}_2 \doteq \langle \langle E, \tau_2 \rangle, p_2, X, \mathfrak{N}_2 \rangle$ be a bundle of Ω -spaces and a map p such that $p = \pi_\mathbf{E} = p_2$ as maps. Thus if the following conditions are satisfied*

- (1) X is compact;
- (2) \mathcal{E} and $\Gamma(p_2)$ are canonically isomorphic as linear spaces

then $\tau(\mathbf{E}, \mathcal{E}) = \tau_2$.

PROOF. By Theorem 5.9. of [Gie] \mathcal{E} and $\Gamma(\pi_\mathbf{E})$ are canonically isomorphic as linear spaces if X is compact, so $\Gamma(\pi_\mathbf{E}) = \Gamma(p_2)$. Moreover $FM(3^*)$ and the shown fact that \mathcal{E} and $\Gamma(\pi_\mathbf{E})$ are canonically isomorphic ensure that $\mathfrak{V}(\mathbf{E}, \mathcal{E})$ is a full bundle, thus it

is so \mathfrak{P}_2 by the equality $\Gamma(\pi_E) = \Gamma(p_2)$. Hence the statement follows by Corollary 1.2.19. \square

1.2.3. Direct Sum of Bundles of Ω -spaces.

DEFINITION 1.2.21 ([Jar]). Let $\{E_i\}_{i \in I}$ a family of lcs. Then we denote by $\tau_0, \tau_b, \tau_l, \tau_l$ respectively the topology on $\bigoplus_{i \in I} E_i$ induced by the product topology on $\prod_{i \in I} E_i$, that induced by the box topology on $\prod_{i \in I} E_i$ (see [Jar]), the direct sum topology, Ch. 4, §3 of [Jar], finally the lc-direct sum topology Ch. 6, §6 of [Jar].

THEOREM 1.2.22. Let $\langle E_i, \nu_i \rangle_{i=1}^n$ a finite family of lcs where $\nu_i = \{\nu_{i,l_i} \mid l_i \in L_i\}$ is a fundamental directed set of seminorms of E_i . Let us set for all $i = 1, \dots, n, l_i \in L_i$ and $\rho \in \prod_{i=1}^n L_i$

$$\begin{cases} \hat{\nu}_{i,l_i} \doteq \nu_{i,l_i} \circ \text{Pr}_i \\ \hat{\mu}_\rho \doteq \sum_{i=1}^n \hat{\nu}_{i,l_i}, \end{cases}$$

where $\text{Pr}_i : \prod_{k=1}^n E_k \ni x \mapsto x_i \in E_i$.

Then $\hat{\mu} \doteq \{\hat{\mu}_\rho \mid \rho \in \prod_{i=1}^n L_i\}$ is a directed set of seminorms on $\bigoplus_{i=1}^n E_i$, moreover by setting

$$\begin{cases} \mathcal{B}(\mathbf{0}) \doteq \{W_\varepsilon^\rho \mid \varepsilon, \rho \in \prod_{i=1}^n L_i\} \\ W_\varepsilon^\rho \doteq \{x \in \bigoplus_{i=1}^n E_i \mid \hat{\mu}_\rho(x) < \varepsilon\}, \end{cases}$$

we have that $\mathcal{B}(\mathbf{0})$ is a base of a filter on $\bigoplus_{i=1}^n E_i$ in addition

$$\mathfrak{F}_{\mathcal{B}(\mathbf{0})}^{\bigoplus_{i=1}^n E_i} = \mathcal{I}_0^\tau,$$

where τ is the unique locally convex topology on $\bigoplus_{i=1}^n E_i$ generated by $\hat{\mu}$ and \mathcal{I}_0^τ is the neighbourhood's filter of $\mathbf{0}$ with respect to the topology τ . Finally with the notations of Definition 1.2.21 we have $\tau = \tau_0 = \tau_b = \tau_l = \tau_l$.

PROOF. Only in this proof we set $I \doteq \{1, \dots, n\}$, $L \doteq \prod_{i \in I} L_i$ and $E^\oplus \doteq \bigoplus_{i=1}^n E_i$. Due to the fact that $n < \infty$ we know that $\prod_{i=1}^n E_i = E^\oplus$ so by [Jar] §4.3. the set $\{\prod_{i=1}^n U_i \mid U_i \in (U)_i\}$ is a $\mathbf{0}$ -basis for the box topology on E^\oplus if \mathfrak{U}_i is a $\mathbf{0}$ -basis for the topology on E_i . Moreover ν_i is directed so by II.3 of [TVS] we can choose

$$(1.2.3) \quad \mathfrak{U}_i = \{V(\nu_{i,l_i}, \varepsilon) \doteq \{x_i \in E_i \mid \nu_{i,l_i}(x_i) < \varepsilon\} \mid \varepsilon > 0, l_i \in L_i\}.$$

Thus if we set

$$(1.2.4) \quad \begin{cases} \mathcal{B}_1(\mathbf{0}) \doteq \{U_\eta^\rho \mid \eta \in (\mathbb{R}_0^+)^n, \rho \in L\} \\ U_\eta^\rho \doteq \{x \in E^\oplus \mid (\forall i \in I)(\hat{\nu}_{i,\rho_i}(x) < \eta_i)\}, \end{cases}$$

then $\mathcal{B}_1(\mathbf{0})$ is a $\mathbf{0}$ –basis for the topology τ_0 . Moreover $U_\varepsilon^\rho = \bigcap_{i=1}^n V(\hat{\nu}_{i,\rho_i}\eta_i)$ so if we set

$$\mathcal{G}(\mathbf{0}) \doteq \left\{ \bigcap_{s \in M} V(\hat{\nu}_s \varepsilon_M(s)) \mid M \in \mathcal{P}_\omega \left(\bigcup_{i \in I} \{i\} \times L_i \right) \varepsilon_M : M \rightarrow \mathbb{R}_0^+ \right\},$$

then by (1.2.4) $\mathcal{B}_1(\mathbf{0}) \subseteq \mathcal{G}(\mathbf{0})$. Moreover by applying II.3 of [TVS], $\mathcal{G}(\mathbf{0})$ is a basis of a filter thus

$$\mathfrak{F}_{\mathcal{B}_1(\mathbf{0})}^{E^\oplus} \subseteq \mathfrak{F}_{\mathcal{G}(\mathbf{0})}^{E^\oplus}.$$

Now for all $M \in \mathcal{P}_\omega \left(\bigcup_{i \in I} \{i\} \times L_i \right)$ we have $M = \bigcup_{i \in I} M_i$ with $M_i \doteq M \cap (\{i\} \times L_i) = \{i\} \times Q_i$ for some $Q_i \in \mathcal{P}_\omega(L_i)$. Hence $\forall M \in \mathcal{P}_\omega \left(\bigcup_{i \in I} \{i\} \times L_i \right)$ and $\forall \varepsilon_M : M \rightarrow \mathbb{R}_0^+$

$$\begin{aligned} T &\doteq \bigcap_{s \in M} V(\hat{\nu}_s, \varepsilon_M(s)) = \bigcap_{i \in I} \bigcap_{s \in M_i} V(\hat{\nu}_s, \varepsilon_M(s)) \\ &= \bigcap_{i \in I} \bigcap_{l_i \in Q_i} \{x \in E^\oplus \mid x_i \in V(\nu_{i,l_i}, \varepsilon_M(i, l_i))\} \\ &= \bigcap_{i \in I} \left\{ x \in E^\oplus \mid x_i \in \bigcap_{l_i \in Q_i} V(\nu_{i,l_i}, \varepsilon_M(i, l_i)) \right\}. \end{aligned}$$

Moreover we know that \mathfrak{U}_i is a basis of a filter on E_i thus for aa $i \in I$ there exists $\lambda_i > 0$ and $k_i \in L_i$ such that

$$V(\nu_{i,k_i}, \lambda_i) \subseteq \bigcap_{l_i \in Q_i} V(\nu_{i,l_i}, \varepsilon_M(i, l_i)),$$

hence

$$\begin{aligned} \mathcal{G}(\mathbf{0}) \ni T &\supseteq \bigcap_{i \in I} \{x \in E^\oplus \mid x_i \in V(\nu_{i,k_i}, \lambda_i)\} \\ &= \bigcap_{i \in I} V(\hat{\nu}_{i,k_i}, \lambda_i) \in \mathcal{B}_1(\mathbf{0}). \end{aligned}$$

Therefore by a well-known property of filters $\mathfrak{F}_{\mathcal{G}(\mathbf{0})}^{E^\oplus} \subseteq \mathfrak{F}_{\mathcal{B}_1(\mathbf{0})}^{E^\oplus}$ then

$$(1.2.5) \quad \mathfrak{F}_{\mathcal{G}(\mathbf{0})}^{E^\oplus} = \mathfrak{F}_{\mathcal{B}_1(\mathbf{0})}^{E^\oplus}.$$

By applying II.3 of [TVS] we know that $\mathfrak{F}_{\mathcal{G}(\mathbf{0})}^{E^\oplus}$ is the $\mathbf{0}$ –neighbourhood's filter with respect to the locally convex topology generated by the family of seminorms $\{\nu_s \mid s \in$

$\bigcup_{i \in I} \{i\} \times L_i$ thus by (1.2.4) and (1.2.5)

$$(1.2.6) \quad \left\{ \nu_s \mid s \in \bigcup_{i \in I} \{i\} \times L_i \right\} \text{ is a fss for } \tau_0,$$

where fss is for fundamental system of seminorms.

Now $\hat{\mu}$ is a set of seminorms on E^\oplus . Let $\rho^1, \rho^2 \in L$ then by the hypothesis that ν_i is directed, for all $i \in I$ there exists $\rho_i \in L_i$ such that $\rho_i \geq \rho^1, \rho^2$ thus $\hat{\mu}_\rho \geq \hat{\mu}_{\rho^1}, \hat{\mu}_{\rho^2}$, hence $\hat{\mu}$ is directed. Therefore setting

$$\begin{cases} \mathcal{B}(\mathbf{0}) \doteq \{W_\varepsilon^\rho \mid \varepsilon > 0, \rho \in L\} \\ W_\varepsilon^\rho \doteq \{x \in E^\oplus \mid \hat{\mu}_\rho(x) < \varepsilon\} \end{cases}$$

by applying II.3 of [TVS]

$$(1.2.7) \quad \mathcal{B}(\mathbf{0}) \text{ is the } \mathbf{0}\text{--basis for the topology gen. by } \hat{\mu}.$$

Now $(\forall (k, l_k) \in \bigcup_{i \in I} \{i\} \times L_i)(\exists \rho \in L)(\hat{\nu}_{k, l_k} \leq a \hat{\mu}_\rho)$ indeed keep any ρ s.t. $\rho(k) = l_k$. While $(\forall \rho \in I)(m \in \mathbb{N})(\exists s_1, \dots, s_m \in \bigcup_{i \in I} \{i\} \times L_i)(\exists a > 0)(\hat{\mu}_\rho \leq a \sup_r \hat{\nu}_{s_r})$ indeed it is sufficient to set $m = n$, $a = n$ and $s_i = (i, \rho_i)$ for all $i \in I$. Therefore by applying Corollary 1 II.7 of [TVS] and by (1.2.7) and (1.2.6) we have that $\hat{\mu}$ is a fundamental directed set of seminorms for the topology τ_0 hence the part of the statement concerning τ_0 . By Prop. 2, §3, Ch 4 of [Jar] we know that $\tau_0 = \tau_b = \tau_l$. Finally $\tau_l = \tau_l$ by the fact that τ_l is the finest locally convex topology among those which are coarser than τ_l , §6, Ch 6 of [Jar], and the just now shown fact that τ_l is locally convex being equal to τ_0 which is generated by $\hat{\mu}$. \square

Now we shall apply the result obtained in the previous proposition, in order to extend to the case of bundles of Ω –spaces what is a standard construction in the Banach bundles case.

DEFINITION 1.2.23. Let $\{\mathfrak{V}_i \doteq \langle \langle \mathfrak{E}_i, \tau_i \rangle, \pi_i, X, \mathfrak{N}_i \rangle\}_{i=1}^n$ be a family of bundles of Ω –spaces, let us denote $\mathfrak{N}_i = \{\nu_{i, l_i} \mid l_i \in L_i\}$ and $\mathfrak{N}_i^x = \{\nu_{i, l_i}^x \doteq \nu_{i, l_i} \upharpoonright (\mathfrak{E}_i)_x \mid l_i \in L_i\}$, with $(\mathfrak{E}_i)_x \doteq \pi_i^{-1}(x)$ for all $i = 1, \dots, n$. Set

- (1) $\mathbf{E}_x^\oplus \doteq \bigoplus_{i=1}^n (\mathfrak{E}_i)_x$.
- (2) $\mathbf{n}_x^\oplus \doteq \{\hat{\mu}_\rho^x \mid \rho \in \prod_{i=1}^n L_i\}$, where

$$(1.2.8) \quad \boxed{\hat{\mu}_\rho^x = \sum_{i=1}^n \hat{\nu}_{i, \rho_i}^x;}$$

(3) \mathcal{E}^\oplus is the linear subspace of $\prod_{x \in X} \mathbf{E}_x^\oplus$ generated by the following set

$$(1.2.9) \quad \bigcup_{i=1}^n \tilde{\Gamma}(\pi_i).$$

Here $\text{Pr}_i^x : \mathbf{E}_x^\oplus \ni x \mapsto x(i) \in (\mathfrak{E}_i)_x$ while $\hat{\nu}_{i,\rho_i}^x = \nu_{i,\rho_i}^x \circ \text{Pr}_i^x$ and $I_i^x : (\mathfrak{E}_i)_x \rightarrow \mathbf{E}_x^\oplus$ is the canonical inclusion, i.e. $\text{Pr}_j^x \circ I_i^x = \delta_{i,j} \text{Id}^x$, finally $\tilde{\Gamma}(\pi_i) \doteq \{\tilde{f} \mid f \in \Gamma(\pi_i)\}$, with $\tilde{f}(x) \doteq I_i^x(f(x))$.

Notice that $\{((\mathfrak{E}_i)_x, \mathfrak{N}_i^x)\}_{i=1}^n$ for all $x \in X$ is a family of Hlcs where \mathfrak{N}_i^x is a directed family of seminorms defining the topology on $(\mathfrak{E}_i)_x$, for all $i = 1, \dots, n$, moreover by Lemma 1.2.25 \mathcal{E}^\oplus satisfies $FM(3) - FM(4)$ with respect to \mathbf{E}^\oplus . Finally by applying Theorem 1.2.22 we have that \mathfrak{n}_x^\oplus is a directed set of seminorms on \mathbf{E}_x^\oplus ⁶ so for what before said we can state that the couple

$$(1.2.10) \quad \langle \mathbf{E}^\oplus, \mathcal{E}^\oplus \rangle,$$

where

$$\mathbf{E}^\oplus \doteq \{ \langle \mathbf{E}_x^\oplus, \mathfrak{n}_x^\oplus \rangle \}_{x \in X},$$

satisfies the requirements of Proposition 5.8. of [Gie]. Therefore (1.2.10) generates a bundle of Ω -spaces which according the notations in Definition is 1.2.5

$$(1.2.11) \quad \bigoplus_{i=1}^n \mathfrak{V}_i \doteq \mathfrak{V}(\mathbf{E}^\oplus, \mathcal{E}^\oplus)$$

and called the bundle direct sum of the family $\{\mathfrak{V}_i\}_{i=1}^n$.

REMARK 1.2.24. Note that Theorem 1.2.22 shows much more than the directness of the set of seminorms \mathfrak{n}_x^\oplus , indeed it proves that \mathfrak{n}_x^\oplus induces on \mathbf{E}_x^\oplus the product topology.

LEMMA 1.2.25. \mathcal{E}^\oplus satisfies $FM(3) - FM(4)$ with respect to \mathbf{E}^\oplus .

PROOF. I_i^x is a bijective map onto its range whose inverse is $\text{Pr}_i^x \upharpoonright \text{Range}(I_i^x)$. Moreover by definition of the product topology Pr_i^x is continuous with respect to the topology τ_0^i on $\text{Range}(I_i^x)$ induced by τ_0 [GT, Ch.1], while I_i^x is continuous with respect to τ_0^i by [Jar, § 4.3 Pr.1] and the definition of τ_l . Hence by Theorem 1.2.22 I_i^x is an isomorphism of the tvs's $\langle (\mathfrak{E}_i)_x, \mathfrak{N}_i^x \rangle$ and $I_i^x((\mathfrak{E}_i)_x)$ as subspace of $\langle \mathbf{E}_x^\oplus, \mathfrak{n}_x^\oplus \rangle$.

⁶ See Remark 1.2.24.

Moreover [Gie, 1.5.III] and [Gie, 1.6.viii]⁷ we deduce that $\{\sigma(x) \mid \sigma \in \Gamma(\pi_i)\}$ is dense in $\langle (\mathfrak{E}_i)_x, \mathfrak{N}_i^x \rangle$. Therefore $\forall i = 1, \dots, n$ and $\forall x \in X$

$$(1.2.12) \quad \{I_i^x(\sigma(x)) \mid \sigma \in \Gamma(\pi_i)\} \text{ is dense in } I_i^x((\mathfrak{E}_i)_x).$$

where $I_i^x((\mathfrak{E}_i)_x)$ has to be intended as topological vector subspace of $\langle \mathbf{E}_x^\oplus, \mathbf{n}_x^\oplus \rangle$. So by the continuity of the sum on $\langle \mathbf{E}_x^\oplus, \mathbf{n}_x^\oplus \rangle$ and the fact that \mathbf{E}_x^\oplus is generated as linear space by the set $\bigcup_{i=1}^n I_i^x((\mathfrak{E}_i)_x)$ we can state $\forall x \in X$ that

$$(1.2.13) \quad \{F(x) \mid F \in \mathcal{E}^\oplus\} \text{ is dense in } \langle \mathbf{E}_x^\oplus, \mathbf{n}_x^\oplus \rangle.$$

Namely by (1.2.12)

$$(\forall v \in \mathfrak{E}^\oplus)(\forall i = 1, \dots, n)(\exists \{\sigma_{\alpha_i}\}_{\alpha_i \in D_i} \text{ net } \subset \Gamma(\pi_i))$$

such that

$$\begin{aligned} v &= \sum_{i=1}^n I_i^x(\text{Pr}_i^x(v)) = \sum_{i=1}^n \lim_{\alpha_i \in D_i} I_i^x(\sigma_{\alpha_i}(x)) \\ &= \sum_{i=1}^n \lim_{\alpha \in D} w_\alpha^i(x) = \lim_{\alpha \in D} \sum_{i=1}^n w_\alpha^i(x) \\ &= \lim_{\alpha \in D} \sum_{i=1}^n I_i^x(\sigma_{\alpha(i)}(x)), \end{aligned}$$

where $D \doteq \prod_{i=1}^n D_i$ while $w_\alpha^i(x) \doteq I_i^x(\sigma_{\alpha(i)}(x))$ for all $\alpha \in D$. Moreover $\forall \alpha \in D$

$$\left(X \ni x \mapsto \sum_{i=1}^n I_i^x(\sigma_{\alpha(i)}(x)) \right) \in \mathcal{E}^\oplus$$

then (1.2.13) and $FM(3)$ follow.

Finally $FM(4)$ follows by [Gie, 1.6.iii] applied to any $\sigma_i \in \Gamma(\pi_i)$ for all $i = 1, \dots, n$ indeed $\forall \sigma_i \in \Gamma(\pi_i)$

$$\nu_{i,\rho_i}^{\hat{x}}(\tilde{\sigma}_i(x)) = \nu_{i,\rho_i}^x \circ \text{Pr}_i^x \circ I_i^x \circ \sigma_i(x) = \nu_{i,\rho_i}^x \circ \sigma_i(x).$$

□

REMARK 1.2.26. By Definition 1.2.5 and (1.2.11)

$$\bigoplus_{i=1}^n \mathfrak{V}_i = \langle \langle \mathfrak{E}(\mathbf{E}^\oplus), \tau(\mathbf{E}^\oplus, \mathcal{E}^\oplus) \rangle, \pi_{\mathbf{E}^\oplus}, X, \mathbf{n}^\oplus \rangle$$

⁷ which ensures that the locally convex topology on $(\mathfrak{E}_i)_x$ generated by the set of seminorms \mathfrak{N}_i^x is exactly the topology induced on it by the topology τ_i on \mathfrak{E}_i , for all i and $x \in X$.

where

- (1) $\mathfrak{E}(\mathbf{E}^\oplus) \doteq \bigcup_{x \in X} \{x\} \times \mathbf{E}_x^\oplus$, $\pi_{\mathbf{E}^\oplus} : \mathfrak{E}(\mathbf{E}^\oplus) \ni (x, v) \mapsto x \in X$.
- (2) $\mathfrak{n}^\oplus = \{\hat{\mu}_\rho : \rho \in \prod_{i=1}^n L_i\}$, with $\hat{\mu}_\rho : \mathfrak{E}(\mathbf{E}^\oplus) \ni (x, v) \mapsto \hat{\mu}_\rho^x(v)$;
- (3) $\tau(\mathbf{E}^\oplus, \mathcal{E}^\oplus)$ is the topology on $\mathfrak{E}(\mathbf{E}^\oplus)$ such that for all $(x, v) \in \mathfrak{E}(\mathbf{E}^\oplus)$

$$\mathcal{I}_{(x,v)}^{\mathfrak{E}(\mathbf{E}^\oplus)} \doteq \mathfrak{F}_{\mathcal{B}((x,v))}^{\mathfrak{E}(\mathbf{E}^\oplus)}$$

is the neighbourhood's filter of (x, v) with respect to it. Here $\mathfrak{F}_{\mathcal{B}((x,v))}^{\mathfrak{E}(\mathbf{E}^\oplus)}$ is the filter on $\mathfrak{E}(\mathbf{E}^\oplus)$ generated by the following filter's base

$$\begin{aligned} \mathcal{B}^\oplus((x, v)) \doteq \{ & T_{\mathbf{E}^\oplus}(U, \sigma, \varepsilon, \rho) \mid U \in \text{Open}(X), \sigma \in \mathcal{E}^\oplus, \varepsilon > 0, \rho \in \prod_{i=1}^n L_i \\ & \mid x \in U, \hat{\mu}_\rho^x(v - \sigma(x)) < \varepsilon \}, \end{aligned}$$

where

$$T_{\mathbf{E}^\oplus}(U, \sigma, \varepsilon, \rho) \doteq \{(y, w) \in \mathfrak{E}(\mathbf{E}^\oplus) \mid y \in U, \hat{\mu}_\rho^y(w - \sigma(y)) < \varepsilon\};$$

Finally the following is a useful characterization of continuous maps valued in $\mathfrak{E}(\mathbf{E}^\oplus)$.

COROLLARY 1.2.27. *Let $\{\mathfrak{V}_i \doteq \langle \langle \mathfrak{E}_i, \tau_i \rangle, \pi_i, X, \mathfrak{N}_i \rangle\}_{i=1}^n$ be a family of bundles of Ω -spaces, $f \in \mathfrak{E}(\mathbf{E}^\oplus)^X$ and $x \in X$. Then f is continuous in x if and only if $f_0^i : X \rightarrow \mathfrak{E}_i$ is continuous in x for all $i = 1, \dots, n$, where $f_0 \in (\bigcup_{z \in X} \mathbf{E}_z^\oplus)^X$ such that $\forall z \in X$ $f(z) = (z, f_0(z))$ and*

$$f_0^i(z) \doteq \text{Pr}_i^{\pi_{\mathbf{E}^\oplus}(f(z))} \circ f_0(z).$$

In particular $f \in \Gamma(\pi_{\mathbf{E}^\oplus})$ if and only if $(X \ni z \mapsto \text{Pr}_i^z \circ f_0(z) \in (\mathfrak{E}_i)_z) \in \Gamma(\pi_i)$, for all $i = 1, \dots, n$.

PROOF. By (1) \Leftrightarrow (5) in Theorem 1.2.9 applied to $\bigcup_{i=1}^n \tilde{\Gamma}(\pi_i)$. □

CONVENTION 1.2.28. *Let $\{\mathfrak{V}_i \doteq \langle \langle \mathfrak{E}_i, \tau_i \rangle, \pi_i, X, \mathfrak{N}_i \rangle\}_{i=1}^n$ be a family of bundles of Ω -spaces, and $\mathfrak{V}(\mathbf{E}^\oplus, \mathcal{E}^\oplus)$ the bundle direct sum of the family $\{\mathfrak{V}_i\}_{i=1}^n$. By construction we have that $\Gamma(\pi_{\mathbf{E}^\oplus}) \subset \prod_{x \in X} \{x\} \times \mathbf{E}_x^\oplus$. In what follows, except contrary mention, we convey to consider with abuse of language in the obvious manner*

$$\Gamma(\pi_{\mathbf{E}^\oplus}) \subset \prod_{x \in X} \bigoplus_{i=1}^n (\mathfrak{E}_i)_x.$$

Similarly for $\Gamma^x(\pi_{\mathbf{E}^\oplus})$ for any $x \in X$. Moreover in the case in which for any $i = 1, \dots, n$ we have $\mathfrak{V}_i = \mathfrak{V}(\mathbf{E}_i, \mathcal{E}_i)$, with obvious meaning of the symbols we consider

$$\Gamma(\pi_{\mathbf{E}^\oplus}) \subset \prod_{x \in X} \bigoplus_{i=1}^n (\mathbf{E}_i)_x .$$

CHAPTER 2

Main Claim

2.1. Map pre-bundle \mathcal{M} relative to a map system

DEFINITION 2.1.1 (Map Systems). $\langle X, \mathbf{E}, \mathcal{S} \rangle$ is a Map System if

- (1) X is a set;
- (2) $\mathbf{E} = \{\langle \mathbf{E}_x, \mathfrak{N}_x \rangle\}_{x \in X}$ is a nice family of Hlcs with $\mathfrak{N}_x \doteq \{\nu_j^x \mid j \in J\}$ for all $x \in X$;
- (3) $(\exists L \neq \emptyset)(\mathcal{S} = \{S_x\}_{x \in X})$ where $S_x \doteq \{B_l^x \mid l \in L\} \subseteq \text{Bounded}(\mathbf{E}_x)$ and $\bigcup_{l \in L} B_l^x$ is total in \mathbf{E}_x for all $x \in X$.

DEFINITION 2.1.2 (Map pre-bundle). We say that \mathbf{M} is a Map pre-bundle relative to $\langle X, Y, \mathbf{E}, \mathcal{S} \rangle$ if

- (1) $\langle X, \mathbf{E}, \mathcal{S} \rangle$ is a map system;
- (2) $\mathbf{M} = \{\langle \mathbf{M}_x, \mathfrak{R}_x \rangle\}_{x \in X}$ is a nice family of Hlcs;
- (3) Y is a Hausdorff topological space and $\forall x \in X$

$$\boxed{\begin{aligned} \mathbf{M}_x &\subseteq \mathcal{C}_c(Y, \mathcal{L}_{S_x}(\mathbf{E}_x)); \\ \mathfrak{R}_x &= \left\{ \sup_{(K,j,l) \in \mathcal{O}} q_{(K,j,l)}^x \upharpoonright \mathbf{M}_x \mid \mathcal{O} \in \mathcal{P}_\omega(\text{Comp}(Y) \times J \times L) \right\}. \end{aligned}}$$

Here we recall that $\mathcal{P}_\omega(A)$ is the class of all finite parts of the set A , $\mathcal{L}_{S_x}(\mathbf{E}_x)$, for all $x \in X$, is the lcs of all continuous linear maps $\mathcal{L}(\mathbf{E}_x)$ on \mathbf{E}_x with the topology of uniform convergence over the sets in S_x , hence its topology is generated by the following set of seminorms

$$(2.1.1) \quad \left\{ p_{j,l}^x : \mathcal{L}(\mathbf{E}_x) \ni \phi \mapsto \sup_{v \in B_l^x} \nu_j^x(\phi(v)) \mid l \in L, j \in J \right\}.$$

Thus by the totality hypothesis and by [TVS, Prop. 3, III.15] $\mathcal{L}_{S_x}(\mathbf{E}_x)$ is Hausdorff. Finally for all $(K, j, l) \in \text{Comp}(Y) \times J \times L$ we set

$$(2.1.2) \quad q_{(K,j,l)}^x : \mathcal{C}_c(Y, \mathcal{L}_{S_x}(\mathbf{E}_x)) \ni f \mapsto \sup_{t \in K} p_{j,l}^x(f(t))$$

REMARK 2.1.3. By the fact that $\{t\}$ is compact for all $t \in Y$ we have that $\bigcup_{K \in \text{Comp}(Y)} K = Y$ thus by the shown fact that $\mathcal{L}_{S_x}(\mathbf{E}_x)$ is Hausdorff we deduce by [GT, Prp. (1), §1.2, Ch 10] that $\mathcal{C}_c(Y, \mathcal{L}_{S_x}(\mathbf{E}_x))$ is Hausdorff. Moreover by [GT, Def. (1), §1.1, Ch 10] and by the fact that (2.1.1) is a f.s.s. on $\mathcal{L}_{S_x}(\mathbf{E}_x)$, we can deduce that $\left\{ \sup_{(K,j,l) \in \mathcal{O}} q_{(K,j,l)}^x \mid \mathcal{O} \in \mathcal{P}_\omega(\text{Comp}(Y) \times J \times L) \right\}$ is a direct f.s.s. on $\mathcal{C}_c(Y, \mathcal{L}_{S_x}(\mathbf{E}_x))$. Hence $\langle \mathbf{M}_x, \mathfrak{R}_x \rangle$ is a topological vector subspace of $\mathcal{C}_c(Y, \mathcal{L}_{S_x}(\mathbf{E}_x))$ so it is Hausdorff, hence by the construction of \mathfrak{R}_x we can state that $\{\langle \mathbf{M}_x, \mathfrak{R}_x \rangle\}_{x \in X}$ is a nice family of Hlcs in agreement with request (2) in Def. 2.1.2.

For the sake of completeness and the large use of it in all the work, we shall use Definition 1.2.5 for giving in the following remark the explicit form of $\mathfrak{V}(\mathbf{M}, \mathcal{M})$.

REMARK 2.1.4. Let $\mathbf{M} = \{\langle \mathbf{M}_x, \mathfrak{R}_x \rangle\}_{x \in X}$ be a map pre-bundle relative to $\langle X, Y, \mathbf{E} = \langle \mathbf{E}_x, \mathfrak{R}_x \rangle_{x \in X}, \mathcal{S} \rangle$, moreover let \mathcal{M} satisfy FM(3) – FM(4) with respect to \mathbf{M} . Let us denote $\mathfrak{R}_x = \{\nu_j^x \mid j \in J\}$ for all $x \in X$ and use the notations in Definition 2.1.2. Thus for the bundle $\mathfrak{V}(\mathbf{M}, \mathcal{M})$ generated by the couple $\langle \mathbf{M}, \mathcal{M} \rangle$ we have

- (1) $\mathfrak{V}(\mathbf{M}, \mathcal{M}) = \langle \langle \mathfrak{E}(\mathbf{M}), \tau(\mathbf{M}, \mathcal{M}) \rangle, \pi_{\mathbf{M}}, X, \mathfrak{R} \rangle$;
- (2) $\mathfrak{E}(\mathbf{M}) \doteq \bigcup_{x \in X} \{x\} \times \mathbf{M}_x$, $\pi_{\mathbf{M}} : \mathfrak{E}(\mathbf{M}) \ni (x, f) \mapsto x \in X$;
- (3) $\mathfrak{R} = \left\{ \sup_{(K,j,l) \in \mathcal{O}} q_{(K,j,l)} \mid \mathcal{O} \in \mathcal{P}_\omega(\text{Comp}(Y) \times J \times L) \right\}$, with $q_{(K,j,l)} : \mathfrak{E}(\mathbf{M}) \ni (x, f) \mapsto q_{(K,j,l)}^x(f)$;
- (4) $\tau(\mathbf{M}, \mathcal{M})$ is the topology on $\mathfrak{E}(\mathbf{M})$ such that for all $(x, f) \in \mathfrak{E}(\mathbf{M})$

$$\mathcal{I}_{(x,f)}^{\mathfrak{E}(\mathbf{M})} \doteq \mathfrak{F}_{\mathbf{B}_{\mathbf{M}}((x,f))}^{\mathfrak{E}(\mathbf{M})}$$

is the neighbourhood's filter of (x, f) with respect to it. Here $\mathfrak{F}_{\mathbf{B}_{\mathbf{M}}((x,f))}^{\mathfrak{E}(\mathbf{M})}$ is the filter on $\mathfrak{E}(\mathbf{M})$ generated by the following filter's base

$$\mathbf{B}_{\mathbf{M}}((x, f)) \doteq \{T_{\mathbf{M}}(U, \sigma, \varepsilon, \mathcal{O}) \mid U \in \text{Open}(X), \sigma \in \mathcal{M}, \varepsilon > 0, \\ \mathcal{O} \in \mathcal{P}_\omega(\text{Comp}(Y) \times J \times L) \mid x \in U, \sup_{(K,j,l) \in \mathcal{O}} q_{(K,j,l)}^x(f - \sigma(x)) < \varepsilon\},$$

where $\forall U \in \text{Open}(X), \sigma \in \mathcal{M}, \varepsilon > 0$ and $\forall \mathcal{O} \in \mathcal{P}_\omega(\text{Comp}(Y) \times J \times L)$

$$T_{\mathbf{M}}(U, \sigma, \varepsilon, \mathcal{O}) \doteq \left\{ (y, g) \in \mathfrak{E}(\mathbf{M}) \mid y \in U, \sup_{(K,j,l) \in \mathcal{O}} q_{(K,j,l)}^y(g - \sigma(y)) < \varepsilon \right\}.$$

In Remark 2.1.4 we gave explicitly the neighbourhood' filters for the topology $\tau(\mathbf{M}, \mathcal{M})$. A simpler form for this filter can be obtained with additional requirement as we showed in Corollary 1.2.18.

REMARK 2.1.5. Let $\mathbf{M} = \{\langle \mathbf{M}_x, \mathfrak{R}_x \rangle\}_{x \in X}$ be a map pre-bundle relative to $\langle X, Y, \mathbf{E} = \langle \mathbf{E}_x, \mathfrak{R}_x \rangle_{x \in X}, \mathcal{S} \rangle$, moreover let \mathcal{M} satisfy $FM(3) - FM(4)$ with respect to \mathbf{M} . Thus by Remark 1.2.7 $\forall U \in \text{Open}(X), \sigma \in \mathcal{M}, \varepsilon > 0$ and $\forall \mathcal{O} \in \mathcal{P}_\omega(\text{Comp}(Y) \times J \times L)$

$$T_{\mathbf{M}}(U, \sigma, \varepsilon, \mathcal{O}) = \bigcup_{y \in U} B_{\mathbf{M}_y, \mathcal{O}, \varepsilon}(\sigma(y))$$

where for all $s \in \mathbf{M}_y$

$$B_{\mathbf{M}_y, \mathcal{O}, \varepsilon}(s) \doteq \left\{ (y, f) \in \mathfrak{E}(\mathbf{M})_y \mid \sup_{(K, j, l) \in \mathcal{O}} q_{(K, j, l)}^y(f - s) < \varepsilon \right\}.$$

By applying Remark 1.2.6 we have the following

REMARK 2.1.6. Let \mathbf{M} be a map pre-bundle relative to $\langle X, Y, \mathbf{E}, \mathcal{S} \rangle$, moreover let \mathcal{M} satisfy $FM(3) - FM(4)$ with respect to \mathbf{M} . Thus

- (1) $\mathfrak{V}(\mathbf{M}, \mathcal{M})$ is a bundle of Ω -spaces;
- (2) with the notations of Definition 2.1.4 $\mathfrak{V}(\mathbf{M}, \mathcal{M})$ is such that
 - (a) $\langle \mathfrak{E}(\mathbf{M})_x, \tau(\mathbf{M}, \mathcal{M}) \rangle$ as topological vector space is isomorphic to $\langle \mathbf{M}_x, \mathfrak{R}_x \rangle$ for all $x \in X$;
 - (b) \mathcal{M} is canonically isomorphic with a linear subspace of $\Gamma(\pi_{\mathbf{M}})$ and if X is compact $\mathcal{M} \simeq \Gamma(\pi_{\mathbf{M}})$.

2.2. (Θ, \mathcal{E}) —structures General case

In Definition 2.2.2 we show how to generalize the topology of uniform convergence to the case of a bundle $\langle \mathfrak{M}, \rho, X \rangle$ of Ω -spaces, such that $\{\mathfrak{M}_x\}_{x \in X}$ is a map pre-bundle relative to $\langle X, Y, \{\mathfrak{E}_x\}_{x \in X}, \mathcal{S} \rangle$, where $\langle \mathfrak{E}, \pi, X \rangle$ is a bundle $\langle \mathfrak{M}, \rho, X \rangle$ of Ω -spaces. The aim is to correlate the topology on \mathfrak{M} with that on \mathfrak{E} in order to generalize the correlation established in the introduction for the trivial bundle case.

DEFINITION 2.2.1.

$$(\bullet) : \prod_{x \in X} (\mathfrak{E}_x)^{\mathfrak{E}_x} \times \prod_{x \in X} \mathfrak{E}_x \rightarrow \prod_{x \in X} \mathfrak{E}_x$$

such that for all $F \in \prod_{x \in X} (\mathfrak{E}_x)^{\mathfrak{E}_x}, v \in \prod_{x \in X} \mathfrak{E}_x$ we have

$$(F \bullet w)(x) \doteq F(x)(w(x)).$$

DEFINITION 2.2.2 ((Θ, \mathcal{E}) —structures). We say that $\langle \mathfrak{V}, \mathfrak{W}, X, Y \rangle$ is a (Θ, \mathcal{E}) —structure if

- (1) $\mathfrak{V} \doteq \langle \langle \mathfrak{E}, \tau \rangle, \pi, X, \mathfrak{N} \rangle$ is a bundle of Ω –spaces;
- (2) $\mathcal{E} \subseteq \Gamma(\pi)$;
- (3) $\Theta \subseteq \prod_{x \in X} \text{Bounded}(\mathfrak{E}_x)$;
- (4) $\forall B \in \Theta$
 - (a) $\mathbf{D}(B, \mathcal{E}) \neq \emptyset$;
 - (b) $\bigcup_{B \in \Theta} \mathcal{B}_B^x$ is total in \mathfrak{E}_x for all $x \in X$;
- (5) $\mathfrak{W} \doteq \langle \langle \mathfrak{M}, \gamma \rangle, \rho, X, \mathfrak{R} \rangle$ is a bundle of Ω –spaces such that $\{\langle \mathfrak{M}_x, \mathfrak{R}_x \rangle\}_{x \in X}$ is a map pre-bundle relative to $\langle X, Y, \{\langle \mathfrak{E}_x, \mathfrak{N}_x \rangle\}_{x \in X}, \mathcal{S} \rangle$.

Here $\mathcal{S} \doteq \{S_x\}_{x \in X}$ and $(\forall B \in \Theta)(\forall x \in X)$

$$(2.2.1) \quad \boxed{\begin{cases} \mathbf{D}(B, \mathcal{E}) \doteq \mathcal{E} \cap \left(\prod_{x \in X} B_x \right) \\ \mathcal{B}_B^x \doteq \{v(x) \mid v \in \mathbf{D}(B, \mathcal{E})\} \\ S_x \doteq \{\mathcal{B}_B^x \mid B \in \Theta\}. \end{cases}}$$

Moreover $\langle \mathfrak{V}, \mathfrak{W}, X, Y \rangle$ is an invariant (Θ, \mathcal{E}) –structure if it is a (Θ, \mathcal{E}) –structure such that

$$(2.2.2) \quad \left\{ F \in \prod_{z \in X}^b \mathfrak{M}_z \mid (\forall t \in Y)(F_t \bullet \mathcal{E}(\Theta) \subseteq \Gamma(\pi)) \right\} = \Gamma(\rho).$$

Finally $\langle \mathfrak{V}, \mathfrak{W}, X, Y \rangle$ is a compatible (Θ, \mathcal{E}) –structure if it is a (Θ, \mathcal{E}) –structure such that for all $t \in Y$

$$(2.2.3) \quad \Gamma(\rho)_t \bullet \mathcal{E}(\Theta) \subseteq \Gamma(\pi).$$

Here

$$\mathcal{E}(\Theta) \doteq \bigcup_{B \in \Theta} \mathbf{D}(B, \mathcal{E}),$$

and $S_t \doteq \{F_t \mid F \in S\}$ and $F_t \in \prod_{x \in X} \mathcal{L}(\mathfrak{E}_x)$ such that $F_t(x) \doteq F(x)(t)$, for all $S \subseteq \prod_{x \in X} \mathcal{L}(\mathfrak{E}_x)^Y$ $t \in Y$, and $F \in S$.

REMARK 2.2.3. Let $\langle \mathfrak{V}, \mathfrak{W}, X, Y \rangle$ be a (Θ, \mathcal{E}) –structure. Then for all $x \in X$

$$(2.2.4) \quad \mathfrak{R}_x = \left\{ \sup_{(K, j, B) \in O} q_{(K, j, B)}^x \upharpoonright \mathfrak{M}_x \mid O \in \mathcal{P}_\omega(\text{Comp}(Y) \times J \times \Theta) \right\}$$

where by using the notations of Def. 2.2.2 we set $\mathfrak{N} = \{\nu_j^x \mid j \in J\}$ and for all $K \in \text{Comp}(Y), j \in J, B \in \Theta$

$$(2.2.5) \quad q_{(K,j,B)}^x : \mathcal{C}_c(Y, \mathcal{L}_{S_x}(\mathfrak{E}_x)) \ni f_x \mapsto \sup_{t \in K} \sup_{v \in \mathbf{D}(B, \mathcal{E})} \nu_j^x(f_x(t)v(x))$$

REMARK 2.2.4. Let $\mathfrak{V} \doteq \langle \langle \mathfrak{E}, \tau \rangle, \pi, X, \mathfrak{N} \rangle$ be a bundle, $\mathbf{M} = \{\langle \mathbf{M}_x, \mathfrak{R}_x \rangle\}_{x \in X}$ a map pre-bundle relative to $\langle X, Y, \{\langle \mathfrak{E}_x, \mathfrak{N}_x \rangle\}_{x \in X}, \mathcal{S} \rangle$ and \mathcal{M} satisfy $FM(3) - FM(4)$ with respect to \mathbf{M} . Then Rmk. 2.1.4 allows us to construct \mathfrak{W} satisfying the condition (5) in Def. 2.2.2. About the problem of satisfying $FM(3) - FM(4)$ recall the very important Rmk. 1.2.2.

The following characterization of $\mathcal{U} \in \Gamma_U^{x_\infty}(\rho)$ will be very important in the sequel.

LEMMA 2.2.5. Let $\langle \mathfrak{V}, \mathfrak{W}, X, Y \rangle$ be a (Θ, \mathcal{E}) -structure, $x_\infty \in W \subseteq X$ and $\mathcal{U} \in \prod_{x \in W}^b \mathfrak{M}_x$. By using the notations in Definition 2.2.2 we have (1) \Leftarrow (2) \Leftarrow (3) \Leftrightarrow (4) moreover if \mathfrak{W} is locally full (1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4), finally if \mathfrak{W} is full we can choose $U = X$ in (2) and $U' = X$ in (3) and (4). Here

$$(2.2.6) \quad \begin{aligned} & (1) \mathcal{U} \in \Gamma_W^{x_\infty}(\rho); \\ & (2) (\exists U \in \text{Op}(X) \mid U \ni x_\infty)(\exists F \in \Gamma_U(\rho))(F(x_\infty) = \mathcal{U}(x_\infty)) \text{ such that } (\forall j \in J)(\forall K \in \text{Compact}(Y))(\forall B \in \Theta) \\ & \quad \lim_{z \rightarrow x_\infty, z \in W \cap U} \sup_{t \in K} \sup_{v \in \mathbf{D}(B, \mathcal{E})} \nu_j(\mathcal{U}(z)(t)v(z) - F(z)(t)v(z)) = 0; \\ & (3) (\exists U' \in \text{Op}(X) \mid U' \ni x_\infty)(\exists F \in \Gamma_{U'}(\rho))(F(x_\infty) = \mathcal{U}(x_\infty)) \text{ and } (\forall U \in \text{Op}(X) \mid U \ni x_\infty)(\forall F \in \Gamma_U(\rho) \mid F(x_\infty) = \mathcal{U}(x_\infty)) \text{ we have (2.2.6)} \\ & \quad (\forall j \in J)(\forall K \in \text{Compact}(Y))(\forall B \in \Theta); \\ & (4) (\exists U' \in \text{Op}(X) \mid U' \ni x_\infty)(\exists F \in \Gamma_{U'}(\rho))(F(x_\infty) = \mathcal{U}(x_\infty)) \text{ and } \mathcal{U} \in \Gamma_W^{x_\infty}(\rho). \end{aligned}$$

PROOF. Apply Corollary 1.2.10 to Definition 2.1.2. □

COROLLARY 2.2.6. Let us assume the hypotheses of Lemma 2.2.5 and that \mathfrak{W} is full. Moreover let $B \in \Theta$ and $v \in \mathbf{D}(B, \mathcal{E})$. Then (1) \Rightarrow (2), where

- (1) $\mathcal{U} \in \Gamma_W^{x_\infty}(\rho)$ and $\exists F \in \Gamma(\rho)$ such that $F(x_\infty) = \mathcal{U}(x_\infty)$ and $(\forall t \in Y)(F(\cdot)(t) \bullet v \in \Gamma(\pi))$;
- (2) $(\forall t \in X)(\mathcal{U}(\cdot)(t) \bullet v \in \Gamma_W^\infty(\pi))$.

PROOF. By the position (1) and by the implication (1) \Rightarrow (3) of Lemma 2.2.5 and by the fact that the union of all compact subsets of Y is Y , being locally compact, we

deduce that $(\exists F \in \Gamma(\rho))(F(x_\infty) = \mathcal{U}(x_\infty))$ such that $(\forall j \in J)(\forall t \in Y)(\forall B \in \Theta)$ and $\forall v \in \mathbf{D}(B, \mathcal{E})$

$$\begin{cases} \lim_{z \rightarrow x_\infty, z \in W} \nu_j (\mathcal{U}(z)(t)v(z) - F(z)(t)v(z)) = 0, \\ F(\cdot)(t) \bullet v \in \Gamma(\pi). \end{cases}$$

Thus the statement follows by implication (3) \Rightarrow (1) of Corollary 1.2.10. \square

In general condition (1) is much more stronger than (2). Let us conclude this section with two results constructing a (Θ, \mathcal{E}) – structure and describing $\Gamma^{x_\infty}(\rho)$ when \mathfrak{V} is trivial.

LEMMA 2.2.7. *Let Z be a normed space X, Y be two topological spaces. Set for all $x \in X$ and $v \in \mathcal{C}_b(X, Z)$*

$$\begin{cases} \mathcal{M} \doteq \{F \in \mathcal{C}_b(X, \mathcal{C}_c(Y, \mathcal{L}_s(Z))) \mid (\forall K \in \text{Comp}(Y)) \\ \quad (C(F, K) \doteq \sup_{(x,s) \in X \times K} \|F(x)(s)\|_{B(Z)} < \infty)\}, \\ \mathbf{M}_x \doteq \overline{\{F(x) \mid F \in \mathcal{M}\}}, \\ \mu_{(v,x)}^K : \mathbf{M}_x \ni G \mapsto \sup_{s \in K} \|G(s)v(x)\|, \\ \mathcal{A}_x \doteq \{\mu_{(w,x)}^K \mid K \in \text{Comp}(Y), w \in \mathcal{C}_b(X, Z)\}, \\ \mathbf{M} \doteq \{\langle \mathbf{M}_x, \mathcal{A}_x \rangle\}_{x \in X}. \end{cases}$$

closure in $\mathcal{C}_c(Y, B_s(Z))$. Then \mathcal{M} satisfies FM3 – FM4 with respect to \mathbf{M}

PROOF. FM(3) is true by construction, let $v \in \mathcal{C}_b(X, Z)$, $K \in \text{Comp}(Y)$, $F \in \mathcal{M}$, then

$$\sup_{x \in X} \mu_{(v,x)}^K(F(x)) \leq \sup_{(x,s) \in X \times K} \|F(x)(s)\|_{B(Z)} \sup_{x \in X} \|v(x)\| < \infty.$$

For all $x, x_0 \in X$

$$(2.2.7) \quad \mu_{(v,x)}^K(F(x)) \leq C\|v(x) - v(x_0)\| + \sup_{s \in K} \|F(x)(s)v(x_0)\|,$$

where $C \doteq \sup_{(x,s) \in X \times K} \|F(x)(s)\|_{B(Z)}$. Moreover the map $\mathcal{C}_c(Y, B_s(Z)) \ni f \mapsto \sup_{s \in K} \|f(s)w\| \in \mathbb{R}^+$, for all $w \in Z$ is a continuous seminorm, hence by the continuity of F also the map $X \ni x \mapsto \sup_{s \in K} \|F(x)(s)w\| \in \mathbb{R}^+$ is continuous. So by (2.2.7) $\overline{\lim}_{x \rightarrow x_0} \mu_{(v,x)}^K(F(x)) \leq \sup_{s \in K} \|F(x_0)(s)v(x_0)\| = \mu_{(v,x_0)}^K(F(x_0))$, and by [GT, (15), §5.6] we have

$$\overline{\lim}_{x \rightarrow x_0} \mu_{(v,x)}^K(F(x)) = \mu_{(v,x_0)}^K(F(x_0)).$$

Therefore by [GT, (13), §5.6], [GT, Prp. 3, §6.2], and the fact that any map g is *u.s.c.* at a point iff $-g$ is *l.s.c.*, we can state that $X \ni x \mapsto \mu_{(v,x)}^K(F(x))$ is *u.s.c.* at x_0 for all $x_0 \in X$, hence it is *u.s.c.*, which is the $FM(4)$ condition. \square

By Lemma 2.2.7 and Def. 1.2.5 we can construct the bundle $\mathfrak{V}(\mathbf{M}, \mathcal{M})$ generated by the couple $\langle \mathbf{M}, \mathcal{M} \rangle$. In the following result I will construct a (Θ, \mathcal{E}) -structure and describe a large subclass of $\Gamma^{x_\infty}(\rho)$.

THEOREM 2.2.8. *Let us assume the notations and hypotheses of Lemma 2.2.7, let \mathfrak{V} be the trivial Banach bundle with constant stalk Z and set $\Theta \doteq \{B_v \mid v \in \mathcal{C}_b(X, Z)\}$. Then*

- (1) $\langle \mathfrak{V}, \mathfrak{V}(\mathbf{M}, \mathcal{M}), X, Y \rangle$ is a $(\Theta, \mathcal{C}_b(X, Z))$ -structure, moreover if X is compact and Y is locally compact then it is compatible;
- (2) Let $f \in \prod_{x \in X} \mathbf{M}_x$ be such that $\sup_{(x,s) \in X \times K} \|f(x)(s)\|_{B(Z)} < \infty$ for all $K \in \text{Comp}(Y)$ then (a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d), where
 - (a) $f \in \Gamma^{x_\infty}(\pi_{\mathbf{M}})$;
 - (b) $(\forall K \in \text{Comp}(Y))(\forall v \in \mathcal{C}_b(X, Z))$

$$\lim_{x \rightarrow x_\infty} \sup_{s \in K} \|f(x)(s)v(x) - f(x_\infty)(s)v(x)\| = 0$$

- (c) $f : X \rightarrow \mathcal{C}_c(Y, B_s(Z))$ continuous at x_∞ ;
- (d) $(\forall K \in \text{Comp}(Y))(\forall w \in Z)$

$$\lim_{x \rightarrow x_\infty} \sup_{s \in K} \|f(x)(s)w - f(x_\infty)(s)w\| = 0.$$

PROOF. By Rmk. 2.2.4 and Lemma 2.2.7 we have that (5) of Def. 2.2.2 follows. $\Gamma(\pi) \simeq \mathcal{C}_b(X, Z)$ hence by Rmk. 3.1.12 the others requests of Def. 2.2.2 follow. Thus the first sentence of statement (1). If X is compact by Lemma 2.2.7 and Rmk. 1.2.6 follows that $\mathcal{M} \simeq \Gamma(\pi_{\mathbf{M}})$, moreover by Rmk. 3.1.12 we have $\mathcal{E}(\Theta) = \mathcal{E}$ and finally $\mathcal{E} \doteq \Gamma(\pi) \simeq \mathcal{C}_b(X, Z)$. Hence the second sentence of statement (1) follows if we show that $\mathcal{M}_t \bullet \mathcal{C}_b(X, Z) \subseteq \mathcal{C}_b(X, Z)$. To this end fix $v \in \mathcal{C}_b(X, Z)$, $F \in \mathcal{M}$, $s \in Y$ and K_s a compact neighbourhood of s , which there exists by the hypothesis that Y is locally compact. Then we have for all $x, x_0 \in X$

$$(2.2.8) \quad \begin{aligned} & \|F(x)(s)v(x) - F(x_\infty)(s)v(x_0)\| \leq \\ & C(F, K_s)\|v(x) - v(x_0)\| + \|(F(x)(s) - F(x_0)(s))v(x_0)\| \end{aligned}$$

By considering that $F \in \mathcal{C}_b(X, \mathcal{C}_c(Y, B_s(Z)))$ and that $s \in K_s$ we have that $\lim_{x \rightarrow x_0} \|(F(x)(s) - F(x_0)(s))v(x_0)\| = 0$. Hence by (2.2.8) we deduce that $F_s \bullet v$

is continuous at x_0 , so continuous on X , in particular X being compact it is also $\|\cdot\|_Z$ -bounded. Thus $F_s \bullet v \in \mathcal{C}_b(X, Z)$ and the second sentence of the statement follows.

Fix $f \in \prod_{x \in X} \mathbf{M}_x$ such that $(\forall K \in \text{Comp}(Y))(\sup_{(x,s) \in X \times K} \|f(x)(s)\|_{B(Z)} < \infty)$. (a) \Leftrightarrow (b) follows by Lemma 2.2.5, the fact that $\mathcal{M} \subseteq \Gamma(\pi_{\mathbf{M}})$ by Rmk. 1.2.6, and by $(H : X \ni x \mapsto f(x_\infty) \in \mathcal{C}_c(Y, B_s(Z))) \in \mathcal{M}$, indeed H it is bounded and continuous being constant, moreover $\sup_{(x,s) \in X \times K} \|H(x)(s)\|_{B(Z)} = \sup_{s \in K} \|f(x_\infty)(s)\|_{B(Z)} < \infty$, for all $K \in \text{Comp}(Y)$. (b) \Rightarrow (d) follows by the fact that $(X \ni x \mapsto w \in Z) \in \mathcal{C}_b(X, Z)$, and (c) \Leftrightarrow (d) is trivial. For all $K \in \text{Comp}(Y)$, $x \in X$ and $s \in K$

$$\begin{aligned} & \| (f(x)(s) - f(x_\infty)(s))v(x) \| \leq \\ & \| f(x)(s)v(x) - f(x_\infty)(s)v(x_\infty) \| + \| f(x_\infty)(s)v(x_\infty) - f(x_\infty)(s)v(x) \| \leq \\ & \| f(x)(s)(v(x) - v(x_\infty)) \| + \| (f(x)(s) - f(x_\infty)(s))v(x_\infty) \| + \| f(x_\infty)(s)(v(x_\infty) - v(x)) \| \leq \\ & (\|f(x)(s)\| + \|f(x_\infty)(s)\|) \|v(x_\infty) - v(x)\| + \| (f(x)(s) - f(x_\infty)(s))v(x_\infty) \| \leq \\ & C(f, K) \|v(x_\infty) - v(x)\| + \| (f(x)(s) - f(x_\infty)(s))v(x_\infty) \|, \end{aligned}$$

where $C(f, K) \doteq \sup_{(x,s) \in X \times K}$. Hence (d) implies (b). \square

DEFINITION 2.2.9. Let $\langle \mathfrak{V}, \mathfrak{W}, X, Y \rangle$ be a (Θ, \mathcal{E}) -structure, $Y_0 \subset Y$ and $\mathcal{V} \in \prod_{x \in X} \mathfrak{M}_x$. We say that \mathcal{V} is *equicontinuous on Y_0* iff $(\forall j \in J)(\exists a > 0)(\exists j_1 \in J)(\forall z \in X)(\forall v_z \in \mathfrak{E}_z)$

$$(2.2.9) \quad \sup_{t \in Y_0} \nu_j(\mathcal{V}(z)(t)v_z) \leq a\nu_{j_1}(v_z)$$

While \mathcal{V} is *equicontinuous* iff it is equicontinuous on Y . Finally \mathcal{V} is *pointwise equicontinuous* iff it is equicontinuous on every point of Y and *compactly equicontinuous* iff it is equicontinuous on every compact of Y . Note that in case \mathfrak{V} is trivial with constant stalk E then \mathcal{V} is equicontinuous on Y_0 if and only if it is equicontinuous in the standard sense the following set of maps $\{\mathcal{V}_0(z)(t) \in \mathcal{L}(E) \mid (z, t) \in X \times Y_0\}$, where $\mathcal{V}_0 \in (\mathcal{L}(E)^Y)^X$ such that $\mathcal{V}(z) = (z, \mathcal{V}_0(z))$ for all $z \in X$.

PROPOSITION 2.2.10. *Let \mathfrak{V} be trivial with costant stalk E , $A^0 \in \text{Bounded}(E)$, $x_\infty \in X$ and ¹*

$$(2.2.10) \quad \begin{cases} \mathcal{E}_0 \subseteq \mathcal{C}_b(X, E) \\ \mathcal{E}_0 \text{ equicontinuous set at } x_\infty \\ \{(X \ni x \mapsto a \in E) \mid a \in A^0\} \subset \mathcal{E}_0. \end{cases}$$

Moreover let $\langle \mathfrak{V}, \mathfrak{W}, X, Y \rangle$ be a (Θ, \mathcal{E}) -structure such that for all $x \in X$

$$\mathfrak{M}_x = \mathcal{C}_c(Y, \mathcal{L}_{S_x}(\{x\} \times E)).$$

and

$$\begin{cases} \mathcal{E} = \prod_{x \in X} \{x\} \times \mathcal{E}_0 \\ \Theta = \{B_{A^0}\} \end{cases}$$

where $B_{A^0}(x) \doteq \{x\} \times A^0$, then

$$(2.2.11) \quad \begin{cases} S_x = \{x\} \times A^0, \forall x \in X \\ \mathfrak{M}_x \simeq \{x\} \times \mathcal{C}_c(Y, \mathcal{L}_{A^0}(E)). \\ \mathfrak{M} = \bigcup_{x \in X} \mathfrak{M}_x \simeq \bigcup_{x \in X} \{x\} \times \mathcal{C}_c(Y, \mathcal{L}_{A^0}(E)) \\ \prod_{x \in X} \mathfrak{M}_x \simeq \prod_{x \in X} \{x\} \times \mathcal{C}_c(Y, \mathcal{L}_{A^0}(E)) \simeq \mathcal{C}_c(Y, \mathcal{L}_{A^0}(E))^X. \end{cases}$$

If \mathfrak{W} is full and

$$\{X \ni x \mapsto \tau_f(x) = (x, f) \in \mathfrak{M}_x \mid f \in \mathcal{C}_c(Y, \mathcal{L}_{A^0}(E))\} \subset \Gamma(\rho),$$

then for all $\mathcal{V} \in \prod_{x \in X}^b \mathfrak{M}_x$, (1) \Rightarrow (2) and (3) \Leftrightarrow (4), where

- (1) $\mathcal{V} \in \Gamma^{x_\infty}(\rho)$
- (2) $\mathcal{V}_0 \in \mathcal{C}(X, \mathcal{C}_c(Y, \mathcal{L}_{A^0}(E)))$,
- (3) \mathcal{V} is compactly equicontinuous and $\mathcal{V} \in \Gamma^{x_\infty}(\rho)$
- (4) \mathcal{V} is compactly equicontinuous and $\mathcal{V}_0 \in \mathcal{C}(X, \mathcal{C}_c(Y, \mathcal{L}_{A^0}(E)))$.

Here in (2) – (4) we consider the isomorphism $\prod_{x \in X} \mathfrak{M}_x \simeq \mathcal{C}_c(Y, \mathcal{L}_{A^0}(E))^X$, and set $\mathcal{V}_0 \in \mathcal{C}_c(Y, \mathcal{L}_{A^0}(E))^X$ such that $\mathcal{V}(x) = (x, \mathcal{V}_0(x))$ for all $x \in X$.

PROOF. For all $x \in X$ by (2.2.1) $\mathcal{B}_{B_{A^0}}^x = \{(x, v_0(x)) \mid v_0 \in \mathcal{E}_0, v_0(X) \subseteq A^0\}$ so $\mathcal{B}_{B_{A^0}}^x \subseteq A^0$. Moreover by construction $(X \ni x \mapsto a \in E) \in \mathcal{E}_0$ for all $a \in A^0$, thus $\mathcal{B}_{B_{A^0}}^x = A^0$. Thus the first equality in (2.2.11) follows, the others are trivial. By

¹See [GT, Def 1, §2.1, Ch. 10] for the definition of equicontinuous sets.

Proposition 1.2.11

$$(2.2.12) \quad (1) \Leftrightarrow \lim_{z \rightarrow x_\infty} \sup_{t \in K} \sup_{v_0 \in \mathcal{E}_0 \cap B_{A^0}} \nu_j ((\mathcal{V}_0(z)(t) - \mathcal{V}_0(x_\infty)(t))v_0(z)) = 0.$$

Moreover by construction we deduce that $\{(X \ni x \mapsto a \in E) \mid a \in A^0\} \subset \mathcal{E}_0 \cap B_{A^0}$, so (2) follows by (1) and (2.2.12). Let $v_0 \in \mathcal{E}_0$ then for all $z \in X$ and $t \in Y$

$$(2.2.13) \quad \begin{aligned} & (\mathcal{V}(z)(t) - \mathcal{V}(x_\infty)(t))v_0(z) = \mathcal{V}(z)(t)(v_0(z) - v_0(x_\infty)) + \\ & (\mathcal{V}(z)(t) - \mathcal{V}(x_\infty)(t))v_0(x_\infty) + \mathcal{V}(x_\infty)(t)(v_0(z) - v_0(x_\infty)). \end{aligned}$$

Moreover by the hypothesis of equicontinuity at x_∞ of the set \mathcal{E}_0 , for all $j \in J$

$$(2.2.14) \quad \lim_{z \rightarrow x_\infty} \sup_{v_0 \in \mathcal{E}_0} \nu_j(v_0(z) - v_0(x_\infty)) = 0.$$

By (2.2.13) and (2.2.9) for all $j \in J$ there exists $j_1 \in J$ and $a > 0$ such that for all $z \in X$

$$(2.2.15) \quad \begin{aligned} & \sup_{t \in K} \sup_{v_0 \in \mathcal{E}_0 \cap B_{A^0}} \nu_j ((\mathcal{V}_0(z)(t) - \mathcal{V}_0(x_\infty)(t))v_0(z)) \leq \\ & 2a \sup_{v_0 \in \mathcal{E}_0 \cap B_{A^0}} \nu_{j_1}(v_0(z) - v_0(x_\infty)) + \\ & \sup_{t \in K} \sup_{v_0 \in \mathcal{E}_0 \cap B_{A^0}} \nu_j (\mathcal{V}(z)(t) - \mathcal{V}(x_\infty)(t)) v_0(x_\infty). \end{aligned}$$

Therefore by (2.2.15), (2.2.14) and by (4) follows

$$\lim_{z \rightarrow x_\infty} \sup_{t \in K} \sup_{v_0 \in \mathcal{E}_0 \cap B_{A^0}} \nu_j ((\mathcal{V}_0(z)(t) - \mathcal{V}_0(x_\infty)(t))v_0(z)) = 0.$$

Hence (1) follows by (2.2.12). □

2.3. Main Claim

The following is the preparatory definition for the first main structure of the paper.

DEFINITION 2.3.1 (*Graph*(($\mathfrak{V}_1, \mathfrak{V}_2$)). Let $\mathfrak{V}_i \doteq \langle \langle \mathfrak{E}_i, \tau_i \rangle, \pi_i, X, \mathfrak{N}_i \rangle_{i=1}^2$ be a couple of bundles of Ω -spaces and with the notations used in Remark 1.2.26 let $\bigoplus_{i=1}^2 \mathfrak{V}_i = \langle \langle \mathfrak{E}(\mathbf{E}^\oplus), \tau(\mathbf{E}^\oplus, \mathcal{E}^\oplus) \rangle, \pi_{\mathbf{E}^\oplus}, X, \mathfrak{n}^\oplus \rangle$ be the bundle direct sum of the family $\{\mathfrak{V}_i\}_{i=1}^2$. Then we say that $\langle \mathcal{T}, x_\infty, \Phi \rangle$ is a graph section or $\langle \mathcal{T}, x_\infty, \Phi \rangle \in \text{Graph}((\mathfrak{V}_1, \mathfrak{V}_2))$ if

- (1) $\mathcal{T} \in \prod_{x \in X} \text{Graph}((\mathfrak{E}_1)_x \times (\mathfrak{E}_2)_x)$;
- (2) $x_\infty \in X$;
- (3)

$$\boxed{\Phi \subseteq \Gamma^{x_\infty}(\pi_{\mathbf{E}^\oplus})}$$

Φ is a linear space such that

Graph inclusion: $(\forall x \in X)(\forall \phi \in \Phi)(\phi(x) \in \mathcal{T}(x))$

Asymptotic Graph:

$$(2.3.1) \quad \boxed{\{\phi(x_\infty) \mid \phi \in \Phi\} = \mathcal{T}(x_\infty)}.$$

DEFINITION 2.3.2 (*Pregraph* $((\mathfrak{V}_1, \mathfrak{V}_2))$). Let $\mathfrak{V}_i \doteq \langle \langle \mathfrak{E}_i, \tau_i \rangle, \pi_i, X, \mathfrak{N}_i \rangle_{i=1}^2$ be a couple of bundles of Ω -spaces and $\bigoplus_{i=1}^2 \mathfrak{V}_i = \langle \langle \mathfrak{E}(\mathbf{E}^\oplus), \tau(\mathbf{E}^\oplus, \mathcal{E}^\oplus) \rangle, \pi_{\mathbf{E}^\oplus}, X, \mathfrak{n}^\oplus \rangle$ the bundle direct sum of the family $\{\mathfrak{V}_i\}_{i=1}^2$. Then we say that $\langle \mathcal{T}_0, x_\infty, \Phi \rangle$ is a pregraph section or $\langle \mathcal{T}_0, x_\infty, \Phi \rangle \in \text{Pregraph}(\mathfrak{V}_1, \mathfrak{V}_2)$ if

- (1) $x_\infty \in X$;
- (2) $\mathcal{T}_0 \in \prod_{x \in X - \{x_\infty\}} \text{Graph}((\mathfrak{E}_1)_x \times (\mathfrak{E}_2)_x)$;
- (3) $\Phi \subseteq \Gamma^{x_\infty}(\pi_{\mathbf{E}^\oplus})$ such that Φ is a linear space and $(\forall x \in X - \{x_\infty\})(\forall \phi \in \Phi)(\phi(x) \in \mathcal{T}_0(x))$.

We shall see in Lemma 3.1.3 that it is possible to construct by any pregraph section $\langle \mathcal{T}_0, x_\infty, \Phi \rangle$ with suitable properties, a corresponding graph section $\langle \mathcal{T}, x_\infty, \Phi \rangle$ such that $\mathcal{T} \upharpoonright (X - \{x_\infty\}) = \mathcal{T}_0$, while $\mathcal{T}(x_\infty)$ is defined by (2.3.1). To this end it is sufficient to show that $\mathcal{T}(x_\infty) \in \text{Graph}(\mathfrak{E}_{x_\infty} \times \mathfrak{E}_{x_\infty})$.

REMARK 2.3.3. Notice that the fundamental requirement that any $\phi \in \Phi$ is a selection continuous in x_∞ implies that

$$\left\{ \begin{array}{l} \{\lim_{z \rightarrow x_\infty} \phi(z) \mid \phi \in \Phi\} = \mathcal{T}(x_\infty) \in \text{Graph}((\mathfrak{E}_1)_{x_\infty} \times (\mathfrak{E}_2)_{x_\infty}) \\ \text{with} \\ \phi(z) \in \mathcal{T}(z) \in \text{Graph}((\mathfrak{E}_1)_z \times (\mathfrak{E}_2)_z), \forall z \in X - \{x_\infty\}, \end{array} \right.$$

which justify the name of asymptotic graph given to (2.3.1). Moreover by setting $X \ni z \mapsto \phi_i(z) \doteq \text{Pr}_i^z(\phi(z))$ we have by Corollary 1.2.27 we have for all $i = 1, 2$

$$(2.3.2) \quad \left\{ \begin{array}{l} \{\lim_{z \rightarrow x_\infty} \phi_i(z) \mid \phi \in \Phi\} = \text{Pr}_i^{x_\infty}(\mathcal{T}(x_\infty)) \text{ with} \\ \phi(z) \in \mathcal{T}(z) \in \text{Graph}((\mathfrak{E}_1)_z \times (\mathfrak{E}_2)_z), \forall z \in X - \{x_\infty\}. \end{array} \right.$$

Finally for $i = 1, 2$ by Corollary 1.2.27 and Corollary 1.2.10 we have $(1_i) \Leftrightarrow (2_i)$

(1_i) : $(\exists \sigma \in \Gamma(\pi))(\sigma(x_\infty) = \phi_i(x_\infty))$ such that

$$(\forall j \in J) \left(\lim_{z \rightarrow x_\infty} \nu_j(\phi_i(z) - \sigma(z)) = 0 \right);$$

(1_{ii}): $(\forall \sigma \in \Gamma(\pi) \mid \sigma(x_\infty) = \phi_i(x_\infty))$ we have

$$(\forall j \in J) \left(\lim_{z \rightarrow x_\infty} \nu_j(\phi_i(z) - \sigma(z)) = 0 \right).$$

DEFINITION 2.3.4 (Class $\Delta \langle \mathfrak{V}, \mathfrak{D}, \Theta, \mathcal{E} \rangle$). Let $\langle \mathfrak{V}, \mathfrak{D}, X, \{pt\} \rangle$ be a (Θ, \mathcal{E}) –structure and let us denote $\mathfrak{D} \doteq \langle \langle \mathfrak{B}, \gamma \rangle, \eta, X, \mathfrak{L} \rangle$. Thus $\Omega \in \Delta \langle \mathfrak{V}, \mathfrak{D}, \Theta, \mathcal{E} \rangle$ if

(1) $\Omega \subseteq \text{Graph}((\mathfrak{V}, \mathfrak{V}))$;

(2) **Projector Section associated to** $\langle \mathcal{T}, x_\infty, \Phi \rangle$: $\forall \langle \mathcal{T}, x_\infty, \Phi \rangle \in \Omega$

$$(2.3.3) \quad \boxed{\left(\exists \mathcal{P} \in \Gamma^{x_\infty}(\eta) \cap \prod_{x \in X} \text{Pr}(\mathfrak{E}_x) \right) (\forall x \in X) (\mathcal{P}(x)T_x \subseteq T_x \mathcal{P}(x))}.$$

Here $T_x : D_x \subseteq \mathfrak{E}_x \rightarrow \mathfrak{E}_x$ is the map such that $\mathcal{T}(x) = \text{Graph}(T_x)$, for all $x \in X$.

CLAIM 2.3.5 (MAIN). Under the assumptions in Definition 2.3.4, possibly with $\langle \mathfrak{V}, \mathfrak{D}, X, \{pt\} \rangle$ invariant, find elements in the class

$$\Delta \langle \mathfrak{V}, \mathfrak{D}, \Theta, \mathcal{E} \rangle.$$

2.4. Induction of Sections of Semigroups

DEFINITION 2.4.1 (Induction of Sections of Semigroups). Let $\langle \mathfrak{V}, \mathfrak{W}, X, \mathbb{R}^+ \rangle$ be a (Θ, \mathcal{E}) –structure such that $\{\mathfrak{E}_x\}_{x \in X}$ is a family of sequentially complete Hlcs and $\mathcal{U}(\mathcal{L}_{S_x}(\mathbf{E}_x)) \subset \mathfrak{M}_x$, where we set $\mathfrak{V} \doteq \langle \langle \mathfrak{E}, \tau \rangle, \pi, X, \mathfrak{N} \rangle$ and $\mathfrak{W} \doteq \langle \langle \mathfrak{M}, \gamma \rangle, \rho, X, \mathfrak{R} \rangle$. Then $\Omega \in \Delta_\Theta \langle \mathfrak{V}, \mathfrak{W}, \mathcal{E}, X, \mathbb{R}^+ \rangle$ if

(1) $\Omega \subseteq \text{Graph}((\mathfrak{E}, \mathfrak{E}))$;

(2) **Semigroup Section associated to** $\langle \mathcal{T}, x_\infty, \Phi \rangle$:

$$\forall \langle \mathcal{T}, x_\infty, \Phi \rangle \in \Omega$$

$$\boxed{\exists \mathcal{U}_{\langle \mathcal{T}, x_\infty, \Phi \rangle} \in \Gamma^{x_\infty}(\rho)}$$

such that $\forall x \in X$

(a) $\mathcal{U}_{\langle \mathcal{T}, x_\infty, \Phi \rangle}(x)$ is an equicontinuous (C_0) –semigroup on \mathfrak{E}_x ;

(b) $(\forall x \in X)(\mathcal{T}(x) = \text{Graph}(R_x))$.

Here R_x is the infinitesimal generator of the semigroup $\mathcal{U}_{\langle \mathcal{T}, x_\infty, \Phi \rangle}(x) \in \mathcal{C}_c(\mathbb{R}^+, \mathcal{L}_{S_x}(\mathfrak{E}_x))$.

CLAIM 2.4.2 (S). Under the assumptions in Definition 2.4.1, possibly with $\langle \mathfrak{V}, \mathfrak{W}, X, \mathbb{R}^+ \rangle$ compatible, find elements in the class

$$\Delta_\Theta \langle \mathfrak{V}, \mathfrak{W}, \mathcal{E}, X, \mathbb{R}^+ \rangle.$$

REMARK 2.4.3. Notice that $\forall \langle \mathcal{T}, x_\infty, \Phi \rangle \in \Omega$ there exists only one semigroup section associated to it. Moreover $\mathcal{U}_{\langle \mathcal{T}, x_\infty, \Phi \rangle}$ is characterized by any of the equivalent conditions in Lemma 2.2.5 with $U = X$ and $Y = \mathbb{R}^+$.

2.5. Induction of a Section of Semigroups - Projectors

DEFINITION 2.5.1 (Induction of a Section of Semigroups - Projectors). Let $\mathfrak{V} \doteq \langle \langle \mathfrak{E}, \tau \rangle, \pi, X, \mathfrak{N} \rangle$, $\mathfrak{D} \doteq \langle \langle \mathfrak{B}, \gamma \rangle, \eta, X, \mathfrak{L} \rangle$ and $\mathfrak{W} \doteq \langle \langle \mathfrak{M}, \gamma \rangle, \rho, X, \mathfrak{R} \rangle$. Moreover let $\langle \mathfrak{V}, \mathfrak{W}, X, \mathbb{R}^+ \rangle$ be a (Θ, \mathcal{E}) -structure such that $\{\mathfrak{E}_x\}_{x \in X}$ is a family of sequentially complete Hlcs and $\langle \mathfrak{V}, \mathfrak{D}, X, \{pt\} \rangle$ be a (Θ, \mathcal{E}) -structure. Then $\Psi \in \Delta_\Theta \langle \mathfrak{V}, \mathfrak{D}, \mathfrak{W}, \mathcal{E}, X, \mathbb{R}^+ \rangle$ if

- (1) $\Psi \subseteq \bigcup_{x_\infty \in X} \Gamma^{x_\infty}(\rho)$;
- (2) $(\forall \mathcal{U} \in \Psi)(\forall x \in X) (\mathcal{U}(x) \text{ is an equicontinuous } (C_0)\text{-semigroup on } \mathfrak{E}_x)$;
- (3) **Projector Section associated to \mathcal{U} :**
 $(\forall x_\infty \in X)(\forall \mathcal{U} \in \Psi \cap \Gamma^{x_\infty}(\rho))$

$$(2.5.1) \quad \left(\exists \mathcal{P} \in \Gamma^{x_\infty}(\eta) \cap \prod_{x \in X} \text{Pr}(\mathfrak{E}_x) \right) (\forall x \in X) (\mathcal{P}(x)H_x \subseteq H_x \mathcal{P}(x)).$$

Here H_x is the infinitesimal generator of the semigroup $\mathcal{U}(x) \in \mathcal{C}_c(\mathbb{R}^+, \mathcal{L}_{S_x}(\mathfrak{E}_x))$ for all $x \in X$.

CLAIM 2.5.2 (S-P). Under the assumptions in Definition 2.4.1, possibly with $\langle \mathfrak{V}, \mathfrak{W}, X, \mathbb{R}^+ \rangle$ compatible and $\langle \mathfrak{V}, \mathfrak{D}, X, \{pt\} \rangle$ invariant, find elements in the class $\Delta_\Theta \langle \mathfrak{V}, \mathfrak{D}, \mathfrak{W}, \mathcal{E}, X, \mathbb{R}^+ \rangle$.

Claims 2.4.2 and 2.5.2 can be used to solve the Main claim 2.3.5 indeed

PROPOSITION 2.5.3. Let $\mathfrak{V} \doteq \langle \langle \mathfrak{E}, \tau \rangle, \pi, X, \mathfrak{N} \rangle$, $\mathfrak{D} \doteq \langle \langle \mathfrak{B}, \gamma \rangle, \eta, X, \mathfrak{L} \rangle$ and $\mathfrak{W} \doteq \langle \langle \mathfrak{M}, \gamma \rangle, \rho, X, \mathfrak{R} \rangle$. Moreover let $\langle \mathfrak{V}, \mathfrak{W}, X, \mathbb{R}^+ \rangle$ be a (Θ, \mathcal{E}) -structure such that $\{\mathfrak{E}_x\}_{x \in X}$ is a family of sequentially complete Hlcs and $\langle \mathfrak{V}, \mathfrak{D}, X, \{pt\} \rangle$ be a (Θ, \mathcal{E}) -structure. Assume that

- (1) $\Omega \in \Delta_\Theta \langle \mathfrak{V}, \mathfrak{W}, \mathcal{E}, X, \mathbb{R}^+ \rangle$;
- (2) $\Psi \in \Delta_\Theta \langle \mathfrak{V}, \mathfrak{D}, \mathfrak{W}, \mathcal{E}, X, \mathbb{R}^+ \rangle$;

and

$$(2.5.2) \quad (\forall \langle \mathcal{T}, x_\infty, \Phi \rangle \in \Omega)(\mathcal{U}_{\langle \mathcal{T}, x_\infty, \Phi \rangle} \in \Psi).$$

Then

$$\Omega \in \Delta \langle \mathfrak{V}, \mathfrak{D}, \Theta, \mathcal{E} \rangle,$$

i.e. Ω satisfies the Main claim 2.3.5. Moreover

$$(\forall \langle \mathcal{T}, x_\infty, \Phi \rangle \in \Omega) (\exists \mathcal{P} \in \Gamma^{x_\infty}(\eta)) (\exists \mathcal{U} \in \Gamma^{x_\infty}(\rho))$$

- (1) $\mathcal{U}(x)$ is an equicontinuous (C_0) -semigroup on \mathfrak{E}_x , for all $x \in X$;
- (2) $(\forall x \in X) (\mathcal{P}(x) \in \text{Pr}(\mathfrak{E}_x))$;
- (3) $(\forall x \in X) (\mathcal{T}(x) = \text{Graph}(R_x))$;
- (4) $\forall x \in X$

$$\mathcal{P}(x)R_x \subseteq R_x\mathcal{P}(x).$$

Here R_x is the infinitesimal generator of the semigroup $\mathcal{U}(x) \in \mathcal{C}_c(\mathbb{R}^+, \mathcal{L}_{S_x}(\mathfrak{E}_x))$, for all $x \in X$.

CHAPTER 3

Semigroup Approximation Theorems

3.1. General Approximation Theorem I

NOTATIONS 3.1.1. For any $\mathfrak{V} \doteq \langle \langle \mathfrak{E}, \tau \rangle, \pi, X, \mathfrak{N} \rangle$ bundle of Ω -spaces and any $\langle \mathcal{T}_0, x_\infty, \Phi \rangle \in \text{Pregraph}((\mathfrak{V}, \mathfrak{V}))$, set $X_0 \doteq X - \{x_\infty\}$, and for any $\phi \in \Phi$ $\phi_i(x) \doteq \text{Pr}_i^x(\phi(x))$ for all $x \in X$ and $i = 1, 2$. Moreover let us denote by T_x the operator in \mathfrak{E}_x such that $\text{Graph}(T_x) = \mathcal{T}_0(x)$, for all $x \in X_0$, while $\mathcal{T} \in \prod_{x \in X} \text{Graph}(\mathfrak{E}_x \times \mathfrak{E}_x)$ so that

$$\begin{cases} \mathcal{T} \upharpoonright X - \{x_\infty\} \doteq \mathcal{T}_0 \\ \mathcal{T}(x_\infty) \doteq \{\phi(x_\infty) \mid \phi \in \Phi\}, \end{cases}$$

in addition set

$$D(T_{x_\infty}) \doteq \overset{x_\infty}{\text{Pr}}_1(\mathcal{T}(x_\infty)) = \{\phi_1(x_\infty) \mid \phi \in \Phi\}.$$

Finally for any map $F : A \rightarrow B$ set $\mathcal{R}(F) \doteq F(A)$ the range of F .

REMARK 3.1.2. Let $\mathfrak{V} \doteq \langle \langle \mathfrak{E}, \tau \rangle, \pi, X, \mathfrak{N} \rangle$ be a bundle of Ω -spaces and $\langle \mathcal{T}_0, x_\infty, \Phi \rangle \in \text{Pregraph}((\mathfrak{V}, \mathfrak{V}))$. By Corollary 1.2.27 $\forall \phi \in \Phi$

$$(3.1.1) \quad \begin{cases} \phi_i \in \Gamma^{x_\infty}(\pi), i = 1, 2 \\ (\forall x \in X_0)(\phi_2(x) = T_x \phi_1(x)). \end{cases}$$

LEMMA 3.1.3. Let $\mathfrak{V} \doteq \langle \langle \mathfrak{E}, \tau \rangle, \pi, X, \mathfrak{N} \rangle$ be a bundle of Ω -spaces, where $\mathfrak{N} \doteq \{\nu_j \mid j \in J\}$. Moreover $\langle \mathcal{T}_0, x_\infty, \Phi \rangle \in \text{Pregraph}((\mathfrak{V}, \mathfrak{V}))$. If for all $x \in X_0$, $v_x \in \text{Dom}(T_x)$, $\lambda > 0$ and $j \in J$ we have $\nu_j((\lambda - T_x)v_x) \geq \lambda \nu_j(v_x)$ and $D(T_{x_\infty})$ is dense in \mathfrak{E}_{x_∞} , then

$$\langle \mathcal{T}, x_\infty, \Phi \rangle \in \text{Graph}((\mathfrak{V}, \mathfrak{V})).$$

Moreover the following

$$(3.1.2) \quad T_{x_\infty} : D(T_{x_\infty}) \ni \phi_1(x_\infty) \mapsto \phi_2(x_\infty)$$

is a well-defined linear operator in \mathfrak{E}_{x_∞} such that $\text{Graph}(T_{x_\infty}) = \mathcal{T}(x_\infty)$ and $\forall v_{x_\infty} \in \text{Dom}(T_{x_\infty}), \forall \lambda > 0$ and $\forall j \in J$ we have

$$\nu_j((\lambda - T_{x_\infty})v_{x_\infty}) \geq \lambda \nu_j(v_{x_\infty}).$$

PROOF. Clearly $\mathcal{T}(x_\infty) \in \text{Graph}(\mathfrak{E}_{x_\infty} \times \mathfrak{E}_{x_\infty})$ if and only if $\phi_1(x_\infty) = \mathbf{0}_{x_\infty}$ implies $\phi_2(x_\infty) = \mathbf{0}_{x_\infty}, \forall \phi \in \Phi$, moreover denoting by T_{x_∞} the corresponding operator we have that $T_{x_\infty} : D(T_{x_\infty}) \rightarrow \mathfrak{E}_{x_\infty}$ is a linear operator. Any real map F defined on a topological space is l.s.c. at a point iff $-F$ is u.s.c. at the same point, see [GT, §6.2. Ch.4], thus by [GT, Prop. 3 §6.2. Ch.4] and [GT, (13), §5.6. Ch.4] $F : X \rightarrow \mathbb{R}$ is u.s.c. in $a \in X$ iff $\overline{\lim}_{x \rightarrow a} F(x) = F(a)$. Moreover by [GT, §6.2. Ch.4] we know that $F : X \rightarrow \overline{\mathbb{R}}$ is l.s.c. at a iff F is continuous at a providing $\overline{\mathbb{R}}$ with the following topology $\{\emptyset, [-\infty, \infty],]a, \infty[\mid a \in \mathbb{R}\}$. Thus for any map $\sigma : Y \rightarrow X$ continuous at b such that $\sigma(b) = a$ we have that $F \circ \sigma$ is l.s.c. at a . Hence because $(-F) \circ \sigma = -(F \circ \sigma)$ we can state that if $F : X \rightarrow \overline{\mathbb{R}}$ is u.s.c. at a then for any map $\sigma : Y \rightarrow X$ continuous at b such that $\sigma(b) = a$ we have that $F \circ \sigma$ is u.s.c. at a . Therefore by using [Gie, 1.6.(ii)] we have $\forall \sigma \in \Gamma^{x_\infty}(\pi)$ and $\forall j \in J$

$$(3.1.3) \quad \nu_j(\sigma(x_\infty)) = \overline{\lim}_{x \rightarrow x_\infty} \nu_j(\sigma(x)).$$

Let $\psi \in \Phi$ such that $\psi_1(x_\infty) = \mathbf{0}_{x_\infty}$ thus $\forall \phi \in \Phi, \forall \lambda > 0, \forall x \in X_0$ and $\forall j \in J$ we have by (3.1.3) and (3.1.1)

$$(3.1.4) \quad \begin{aligned} & \nu_j(\lambda \phi_1(x_\infty) - \phi_2(x_\infty) - \lambda \psi_2(x_\infty)) = \\ & \overline{\lim}_{x \rightarrow x_\infty} \nu_j((\lambda - T_x)(\phi_1(x) + \lambda \psi_1(x))) \geq \\ & \overline{\lim}_{x \rightarrow x_\infty} \lambda \nu_j(\phi_1(x) + \lambda \psi_1(x)) = \lambda \nu_j(\phi_1(x_\infty)), \end{aligned}$$

where, the inequality comes by [GT, Prop. 11 §5.6. Ch.4] and by the hypothesis $\nu_j((\lambda - T_x)(\phi_1(x) + \lambda \psi_1(x))) \geq \lambda \nu_j(\phi_1(x) + \lambda \psi_1(x))$ for all $x \in X_0$. Now $\lim_{\lambda \rightarrow \infty} v/\lambda = \mathbf{0}_{x_\infty}$ for any $v \in \mathfrak{E}_{x_\infty}$, hence by the fact that $\nu_j^{x_\infty} \doteq \nu_j \upharpoonright \mathfrak{E}_{x_\infty}$ is a continuous seminorm and by (3.1.4) $(\forall j \in J)(\forall \phi \in \Phi)$

$$(3.1.5) \quad \nu_j(\phi_1(x_\infty) - \psi_2(x_\infty)) = \lim_{\lambda \rightarrow \infty} \frac{\nu_j(\lambda \phi_1(x_\infty) - \phi_2(x_\infty) - \lambda \psi_2(x_\infty))}{\lambda} \geq \nu_j(\phi_1(x_\infty)).$$

By hypothesis $D(T_{x_\infty}) = \{\phi_1(x_\infty) \mid \phi \in \mathcal{T}(x_\infty)\}$ is dense in \mathfrak{E}_{x_∞} thus $\nu_j(\psi_2(x_\infty)) = 0$ for all $j \in J$. Indeed let $j \in J$ and $v \in \mathfrak{E}_{x_\infty}$ thus $\exists \{\phi^\alpha\}_{\alpha \in D}$ net in Φ such that $\lim_{\alpha \in D} \phi_1^\alpha(x_\infty) = v$ in \mathfrak{E}_{x_∞} . So by the continuity of $\nu_j^{x_\infty}$ and by (3.1.5) we have $\forall v \in$

\mathfrak{E}_{x_∞}

$$\nu_j(v - \psi_2(x_\infty)) = \lim_{\alpha \in D} \nu_j(\phi_1^\alpha(x_\infty) - \psi_2(x_\infty)) \geq \lim_{\alpha \in D} \nu_j(\phi_1^\alpha(x_\infty)) = \nu_j(v).$$

True in particular for $v = 3\psi_2(x_\infty)$, which implies $\nu_j(\psi_2(x_\infty)) = 0$. Hence $\psi_2(x_\infty) = 0_{x_\infty}$ because of \mathfrak{E}_{x_∞} is a Hausdorff lcs for which $\{\nu_j^{x_\infty}\}_{j \in J}$ is a generating set of seminorms of its topology. Thus T_{x_∞} is a well-defined (necessarily linear) operator in \mathfrak{E}_{x_∞} and consequently $\langle \mathcal{T}, x_\infty, \Phi \rangle \in \text{Graph}((\mathfrak{V}, \mathfrak{V}))$. Finally $(\forall j \in J)(\forall \phi \in \Phi)(\forall \lambda > 0)$

$$\begin{aligned} \nu_j((\lambda - T_{x_\infty})\phi_1(x_\infty)) &= \\ \nu_j(\lambda\phi_1(x_\infty) - \phi_2(x_\infty)) &= \quad \text{by (3.1.1), (3.1.3)} \\ \overline{\lim}_{x \rightarrow x_\infty} \nu_j(\lambda\phi_1(x) - \phi_2(x)) &= \quad \text{by (3.1.1)} \\ \overline{\lim}_{x \rightarrow x_\infty} \nu_j((\lambda - T_x)\phi_1(x)) &\geq \quad \text{by hypoth. and [GT, Prop. 11 §5.6. Ch.4]} \\ \overline{\lim}_{x \rightarrow x_\infty} \nu_j(\lambda\phi_1(x)) &= \nu_j(\lambda\phi_1(x_\infty)). \end{aligned}$$

□

LEMMA 3.1.4. *In addition to the hypotheses and notations of Lemma 3.1.3 assume that*
 $(\forall x \in X_0)(\forall \lambda \in \mathbb{R})(\forall j \in J)(\forall v_x \in \text{Dom}(T_x))$

$$(3.1.6) \quad \nu_j((\mathbf{1} - \lambda T_x)v_x) \geq \nu_j(v_x).$$

Thus $(\forall \lambda \in \mathbb{R})(\forall j \in J)(\forall v_{x_\infty} \in \text{Dom}(T_{x_\infty}))$

$$(3.1.7) \quad \nu_j((\mathbf{1} - \lambda T_{x_\infty})v_{x_\infty}) \geq \nu_j(v_{x_\infty}).$$

Moreover $\forall \lambda \in \mathbb{R}$

$$(3.1.8) \quad \begin{cases} \exists (\mathbf{1} - \lambda T_{x_\infty})^{-1} \in \mathcal{L}(\mathcal{R}(\mathbf{1} - \lambda T_{x_\infty}), \mathfrak{E}_{x_\infty}), \\ (\forall w \in \mathcal{R}(\mathbf{1} - \lambda T_{x_\infty}))(\forall j \in J)\nu_j((\mathbf{1} - \lambda T_{x_\infty})^{-1}w) \leq \nu_j(w). \end{cases}$$

Finally

$$(3.1.9) \quad \mathcal{R}(\mathbf{1} - \lambda T_{x_\infty}) \text{ is closed in } \mathfrak{E}_{x_\infty}.$$

PROOF. $(\forall j \in J)(\forall \phi \in \Phi)(\forall \lambda \in \mathbb{R})$

$$\begin{aligned}
& \nu_j((\mathbf{1} - \lambda T_{x_\infty})\phi_1(x_\infty)) = \\
& \nu_j(\phi_1(x_\infty) - \lambda \phi_2(x_\infty)) = \quad \text{by (3.1.1), (3.1.3)} \\
& \overline{\lim}_{x \rightarrow x_\infty} \nu_j(\phi_1(x) - \lambda \phi_2(x)) = \quad \text{by (3.1.1)} \\
& \overline{\lim}_{x \rightarrow x_\infty} \nu_j((\mathbf{1} - \lambda T_x)\phi_1(x)) \geq \quad \text{by (3.1.6) and [GT, Prop. 11 §5.6. Ch.4]} \\
& \overline{\lim}_{x \rightarrow x_\infty} \nu_j(\phi_1(x)) = \nu_j(\phi_1(x_\infty)).
\end{aligned}$$

thus (3.1.7) follows. Let $\lambda \in \mathbb{R}$, by (3.1.7) we obtain (3.1.8), indeed $\forall f, g \in \text{Dom}(T_{x_\infty})$ such that $(\mathbf{1} - \lambda T_{x_\infty})f = (\mathbf{1} - \lambda T_{x_\infty})g$ we have $\forall j \in J$

$$0 = \nu_j((\mathbf{1} - \lambda T_{x_\infty})(f - g)) \geq \nu_j(f - g),$$

so $f = g$ because of by construction \mathfrak{E}_{x_∞} is Hausdorff. Thus the following is a well-set map

$$(\mathbf{1} - \lambda T_{x_\infty})^{-1} : \mathcal{R}(\mathbf{1} - \lambda T_{x_\infty}) \ni (\mathbf{1} - \lambda T_{x_\infty})f \mapsto f \in \mathfrak{E}_{x_\infty},$$

moreover by (3.1.7) we obtain the second sentence of (3.1.8), hence the first one follows by the fact that the inverse map of any linear operator is linear. By (3.1.8), [GT, Prop. 3 §3.1. Ch.3] and [GT, Prop. 11 §3.6. Ch.2] we deduce that

$$(3.1.10) \quad (\exists ! B \in \mathcal{L}(\overline{\mathcal{R}(\mathbf{1} - \lambda T_{x_\infty})}, \mathfrak{E}_{x_\infty}))(B \upharpoonright \mathcal{R}(\mathbf{1} - \lambda T_{x_\infty}) = (\mathbf{1} - \lambda T_{x_\infty})^{-1}).$$

Let $w \in \overline{\mathcal{R}(\mathbf{1} - \lambda T_{x_\infty})}$ thus $\exists \{f_\alpha\}_{\alpha \in D}$ net in $\text{Dom}(T_{x_\infty})$ such that

$$(3.1.11) \quad w = \lim_{\alpha \in D} (\mathbf{1} - \lambda T_{x_\infty})f_\alpha,$$

therefore by (3.1.10)

$$(3.1.12) \quad Bw = \lim_{\alpha \in D} f_\alpha,$$

while by (3.1.11) and (3.1.12)

$$\begin{aligned}
w - Bw &= \lim_{\alpha \in D} ((f_\alpha - \lambda T_{x_\infty} f_\alpha) - f_\alpha) \\
&= \lim_{\alpha \in D} -\lambda T_{x_\infty} f_\alpha.
\end{aligned}$$

So

$$(3.1.13) \quad Bw - w = \lim_{\alpha \in D} \lambda T_{x_\infty} f_\alpha.$$

By (3.1.12), (3.1.13) and the fact that λT_{x_∞} is closed, we obtain

$$\begin{cases} Bw \in \text{Dom}(T_{x_\infty}), \\ \lambda T_{x_\infty}(Bw) = Bw - w, \end{cases}$$

which means $w = (\mathbf{1} - \lambda T_{x_\infty})Bw$, so $w \in \mathcal{R}(\mathbf{1} - \lambda T_{x_\infty})$ and (3.1.9) follows. \square

LEMMA 3.1.5. *Let us assume the hypotheses of Lemma 3.1.4, moreover let $\lambda \in \mathbb{R} - \{0\}$, $\{\lambda_n\}_{n \in \mathbb{N}} \subset \mathbb{R} - \{0\}$ such that $\lim_{n \in \mathbb{N}} \lambda_n = \lambda$. Thus*

$$\bigcap_{n \in \mathbb{N}} \mathcal{R}(\mathbf{1} - \lambda_n T_{x_\infty}) \subseteq \mathcal{R}(\mathbf{1} - \lambda T_{x_\infty}).$$

PROOF. Set only in this proof $T \doteq T_{x_\infty}$. Let $n \in \mathbb{N}$, by (3.1.8) $\exists (\mathbf{1} - \lambda_n T)^{-1} : \mathcal{R}(\mathbf{1} - \lambda_n T_{x_\infty}) \rightarrow \text{Dom}(t)$ moreover

$$\begin{cases} \mathbf{1} - \lambda T = \lambda(\lambda^{-1} - T), \\ (\mathbf{1} - \lambda_n T)^{-1} = \lambda_n^{-1}(\lambda_n^{-1} - T)^{-1}. \end{cases}$$

Let $g \in \bigcap_{n \in \mathbb{N}} \mathcal{R}(\mathbf{1} - \lambda_n T_{x_\infty})$ thus

$$\begin{aligned} (\mathbf{1} - \lambda T)(\mathbf{1} - \lambda_n T)^{-1}g - g &= \frac{\lambda}{\lambda_n}(\lambda^{-1} - T)(\lambda_n^{-1} - T)^{-1}g - g \\ &= \frac{\lambda}{\lambda_n}(\lambda^{-1}(\lambda_n^{-1} - T)^{-1}g - \lambda_n^{-1}(\lambda_n^{-1} - T)^{-1}g) \\ &= \frac{\lambda}{\lambda_n}(\lambda^{-1} - \lambda_n^{-1})(\lambda_n^{-1} - T)^{-1}g, \end{aligned}$$

where in the second equality we considered that $-T(\lambda_n^{-1} - T)^{-1}g - g = -\lambda_n^{-1}(\lambda_n^{-1} - T)^{-1}g$ obtained by $(\lambda_n^{-1} - T)(\lambda_n^{-1} - T)^{-1}g = g$. Thus $\forall j \in J$ by (3.1.8)

$$\nu_j((\mathbf{1} - \lambda T)(\mathbf{1} - \lambda_n T)^{-1}g - g) \leq \left| \frac{\lambda}{\lambda_n} \right| |\lambda^{-1} - \lambda_n^{-1}| \nu_j(g).$$

But $\lim_{n \in \mathbb{N}} |\lambda^{-1} - \lambda_n^{-1}| = 1$ and $\lim_{n \in \mathbb{N}} |\lambda^{-1} - \lambda_n^{-1}| = 0$ so $\nu_j((\mathbf{1} - \lambda T)(\mathbf{1} - \lambda_n T)^{-1}g - g) = 0$, for all $j \in J$. Therefore

$$\lim_{n \in \mathbb{N}} (\mathbf{1} - \lambda T)(\mathbf{1} - \lambda_n T)^{-1}g = g,$$

and the statement follows by (3.1.9). \square

LEMMA 3.1.6. *Under the hypotheses and notations of Lemma 3.1.3 we have that $\mathbf{1} - \lambda T_{x_\infty}$ is a closed operator.*

PROOF. Let $(a, b) \in \overline{Graph(\mathbf{1} - \lambda T_{x_\infty})}$ closure in the space $\mathfrak{E}_{x_\infty} \times \mathfrak{E}_{x_\infty}$ with the product topology. Thus $(\forall \varepsilon > 0)(\forall j \in J)(\exists v_{(\varepsilon, j)} \in Dom(T_{x_\infty}))$

$$\begin{cases} \nu_j(a - v_{(\varepsilon, j)}) < \frac{\varepsilon}{2}, \\ \nu_j(b - (\mathbf{1} - \lambda T_{x_\infty})v_{(\varepsilon, j)}) < \frac{\varepsilon}{2}, \end{cases}$$

so

$$\nu_j((b - a) + \lambda T_{x_\infty} v_{(\varepsilon, j)}) \leq \nu_j(b - (\mathbf{1} - \lambda T_{x_\infty})v_{(\varepsilon, j)}) + \nu_j(a - v_{(\varepsilon, j)}) \leq \varepsilon.$$

Therefore $(\forall \varepsilon > 0)(\forall j \in J)(\exists v_{(\varepsilon, j)} \in Dom(T_{x_\infty}))$

$$\begin{cases} \nu_j(a - v_{(\varepsilon, j)}) < \varepsilon, \\ \nu_j((b - a) - (-\lambda T_{x_\infty} v_{(\varepsilon, j)})) < \varepsilon, \end{cases}$$

which means $(a, (b - a)) \in \overline{Graph(-\lambda T_{x_\infty})}$. Moreover $-\lambda T_{x_\infty}$ is a closed operator thus $b - a = -\lambda T_{x_\infty} a$ or equivalently $(a, b) \in Graph(\mathbf{1} - \lambda T_{x_\infty})$. \square

REMARK 3.1.7. By (3.1.1) we have $\forall \phi \in \Phi$ that $\phi_1(x_\infty) = \lim_{z \rightarrow x_\infty} \phi_1(z)$ and $\phi_2(x_\infty) = \lim_{z \rightarrow x_\infty} \phi_2(z) = \lim_{z \rightarrow x_\infty} T_x \phi_1(z)$, hence

$$\begin{cases} \phi_1(x_\infty) = \lim_{z \rightarrow x_\infty} \phi_1(z) \\ T_{x_\infty} \phi_1(x_\infty) = \lim_{z \rightarrow x_\infty} T_z \phi_1(z). \end{cases}$$

DEFINITION 3.1.8. Let $\lambda \in \mathbb{R}^+$ set

$$\mu_\lambda : \mathcal{C}_{cs}(\mathbb{R}^+, \mathbb{R}) \ni f \mapsto \int_{\mathbb{R}^+} e^{-s\lambda} f(s) ds,$$

where the integral is with respect to the Lebesgue measure on \mathbb{R}^+ .

DEFINITION 3.1.9. Let $\mathfrak{W} \doteq \langle \langle \mathfrak{M}, \gamma \rangle, \rho, X, \mathfrak{R} \rangle$ and $\langle \mathfrak{V}, \mathfrak{W}, X, \mathbb{R}^+ \rangle$ be a (Θ, \mathcal{E}) -structure such that for all $x \in X$

$$(3.1.14) \quad \mathfrak{M}_x \subseteq \bigcap_{\lambda > 0} \mathfrak{L}_1(\mathbb{R}^+, \mathcal{L}_{S_x}(\mathfrak{E}_x); \mu_\lambda),$$

about S_x and \mathfrak{E}_x see Definition 2.2.2. Let $x \in X$, $\mathcal{O} \subseteq \Gamma(\rho)$. and $\mathcal{D} \subseteq \Gamma(\pi)$. By recalling Def. 1.1.1, we say that $\langle \mathfrak{V}, \mathfrak{W}, X, \mathbb{R}^+ \rangle$ has the Laplace duality property on \mathcal{O} and \mathcal{D} at x , shortly $\mathbf{LD}_x(\mathcal{O}, \mathcal{D})$ if

$$(\forall \lambda > 0)(\mathfrak{L}(\Gamma_{\mathcal{O}}^x(\rho))_\lambda \bullet \Gamma_{\mathcal{D}}^x(\pi) \subseteq \Gamma^x(\pi)).$$

Moreover we say that $\langle \mathfrak{V}, \mathfrak{W}, X, \mathbb{R}^+ \rangle$ has the full Laplace duality property on \mathcal{O} and \mathcal{D} , shortly **LD**(\mathcal{O}, \mathcal{D}) if

$$(\forall \lambda > 0)(\mathfrak{L}(\mathcal{O})_\lambda \bullet \mathcal{D} \subseteq \Gamma(\pi)).$$

Finally **LD** is for **LD**($\Gamma(\rho), \Gamma(\pi)$). Here $\mathfrak{L} : \prod_{x \in X} \mathfrak{M}_x \rightarrow \prod_{x \in X} \mathcal{L}_{S_x}(\mathfrak{E}_x)^{\mathbb{R}^+}$ such that $(\forall x \in X)(\forall \lambda \in \mathbb{R}^+)$

$$\mathfrak{L}(F)(x)(\lambda) \doteq \int_0^\infty e^{-\lambda s} F(x)(s) ds,$$

where we recall that the integration is with respect to the Lebesgue measure on \mathbb{R}^+ and with respect to the lct on $\mathcal{L}_{S_x}(\mathfrak{E}_x)$. Finally we used the notations in Def 1.1.1.

REMARK 3.1.10. Under the notations and assumptions of Def. 3.1.9 we have

$$\begin{cases} \mathfrak{L}(\Gamma_{\mathcal{O}}^x(\rho)) \subseteq \Gamma_{\mathcal{O}}^x(\rho) \\ (\forall t > 0)(\Gamma_{\mathcal{O}}^x(\rho)_t \bullet \Gamma_{\mathcal{D}}^x(\pi) \subseteq \Gamma^x(\pi)) \end{cases} \Rightarrow \langle \mathfrak{V}, \mathfrak{W}, X, \mathbb{R}^+ \rangle \text{ has the } \mathbf{LD}_x(\mathcal{O}, \mathcal{D}).$$

Similarly

$$\begin{cases} \mathfrak{L}(\mathcal{O}) \subseteq \mathcal{O} \\ (\forall t > 0)(\mathcal{O}_t \bullet \mathcal{D} \subseteq \Gamma(\pi)) \end{cases} \Rightarrow \langle \mathfrak{V}, \mathfrak{W}, X, \mathbb{R}^+ \rangle \text{ has the } \mathbf{LD}(\mathcal{O}, \mathcal{D}).$$

A useful property is the following one

PROPOSITION 3.1.11. Let $\langle \mathfrak{V}, \mathfrak{W}, X, \mathbb{R}^+ \rangle$ be a (Θ, \mathcal{E}) -structure satisfying (3.1.14), $x_\infty \in X$. Set $S_z = \{B_l^z \mid l \in L\}$, then $\forall z \in X, \forall G \in \mathfrak{L}_1(\mathbb{R}^+, \mathcal{L}_{S_z}(\mathfrak{E}_z); \mu_\lambda)$ and $\forall w_z \in \bigcup_{l \in L} B_l^z$

$$(3.1.15) \quad \left(\int_0^\infty e^{-\lambda s} G(s) ds \right) w_z = \int_0^\infty e^{-\lambda s} G(s) w_z ds.$$

Here in the second member the integration is with respect to the lct on \mathfrak{E}_z , while in the first member the integration is with respect to the lct on $\mathcal{L}_{S_x}(\mathfrak{E}_x)$.

PROOF. Let $z \in X$ and $v \in \bigcup_{l \in L} B_l^z = \mathfrak{E}_z$ then map $\mathcal{L}_{S_z}(\mathfrak{E}_z) \ni A \mapsto Av \in \mathfrak{E}_z$ is linear and continuous. Indeed let $l(v) \in L$ such that $v \in B_{l(v)}^z$, thus we have $\nu_j^z(Av) \leq \sup_{w \in B_{l(v)}^z} \nu_j^z(Aw) \doteq p_{j, l(v)}^z(A)$. Hence by a well-known result in vector valued integration we have (3.1.15). \square

REMARK 3.1.12. Let $\mathfrak{V} \doteq \langle \langle \mathfrak{E}, \tau \rangle, \pi, X, \mathfrak{N} \rangle$ be a bundle of Ω -spaces and $\mathcal{E} \subseteq \prod_{x \in X} \mathfrak{E}_x$. Set for all $v \in \prod_{x \in X} \mathfrak{E}_x$

$$(3.1.16) \quad \begin{cases} B_v : X \ni x \mapsto \{v(x)\}, \\ \Theta \doteq \{B_w \mid w \in \mathcal{E}\} \end{cases}$$

Thus $\Theta \subset \prod_{x \in X} \text{Bounded}(\mathfrak{E}_x)$ and $\forall v \in \mathcal{E}$

$$(3.1.17) \quad \mathcal{E} \cap B_v = \{v\}.$$

Therefore for all $v \in \mathcal{E}$, and for all $x \in X$ with the notations of Def. 2.2.2

$$\begin{cases} \mathbf{D}(B_v, \mathcal{E}) = \{v\}, \\ \mathcal{B}_{B_v}^x = \{v(x)\}, \\ S_x = \{\{w(x)\} \mid w \in \mathcal{E}\}, \\ \mathcal{E}(\Theta) = \mathcal{E}. \end{cases}$$

DEFINITION 3.1.13. Let $\mathfrak{V} \doteq \langle \langle \mathfrak{E}, \tau \rangle, \pi, X, \|\cdot\| \rangle$ be a Banach bundle. Let $x_\infty \in X$ and $\mathcal{U}_0 \in \prod_{x \in X_0} \mathcal{C}(\mathbb{R}^+, B_s(\mathfrak{E}_x))$ be such that $\mathcal{U}_0(x)$ is a (C_0) -semigroup of contractions (respectively of isometries) on \mathfrak{E}_x for all $x \in X_0$. Moreover let us denote by T_x the infinitesimal generator of the semigroup $\mathcal{U}_0(x)$ for any $x \in X_0$ and set

$$(3.1.18) \quad \begin{cases} \mathcal{T}_0(x) \doteq \text{Graph}(T_x), x \in X_0 \\ \Phi \doteq \{\phi \in \Gamma^{x_\infty}(\pi_{\mathbf{E}^\oplus}) \mid (\forall x \in X_0)(\phi(x) \in \mathcal{T}_0(x))\} \\ \mathcal{E} \doteq \{v \in \Gamma(\pi) \mid (\exists \phi \in \Phi)(v(x_\infty) = \phi_1(x_\infty))\} \\ \Theta \doteq \{B_w \mid w \in \mathcal{E}\}, \end{cases}$$

where $\langle \langle \mathfrak{E}(\mathbf{E}^\oplus), \tau(\mathbf{E}^\oplus, \mathcal{E}^\oplus) \rangle, \pi_{\mathbf{E}^\oplus}, X, \mathbf{n}^\oplus \rangle$ is the bundle direct sum of the family $\{\mathfrak{V}, \mathfrak{V}\}$.

The following is a direct generalization of the definition given in [Kur, Lm. 2.11]

DEFINITION 3.1.14. Let $\mathfrak{V} \doteq \langle \langle \mathfrak{E}, \tau \rangle, \pi, X, \mathfrak{N} \rangle$ be a bundle of Ω -spaces, where $\mathfrak{N} \doteq \{\nu_j \mid j \in J\}$. Moreover let Y be a topological space, $s_0 \in Y$, $f \in \prod_{x \in X} \mathfrak{E}_x^Y$ and $\{z_n\}_{n \in \mathbb{N}} \subset X$. Then we say that $\{f(z_n)\}_{n \in \mathbb{N}}$ is bounded if $\sup_{(n,s) \in \mathbb{N} \times Y} \nu_j(f(z_n)(s)) < \infty$ for all $j \in J$. $\{f(z_n)\}_{n \in \mathbb{N}}$ is equicontinuous at s_0 if for all $j \in J$ and for all $\varepsilon > 0$ there exists a neighbourhood U of s_0 such that for all $s \in U$ we have $\sup_{n \in \mathbb{N}} \nu_j(f(z_n)(s) - f(z_n)(s_0)) \leq \varepsilon$. Finally $\{f(z_n)\}_{n \in \mathbb{N}}$ is equicontinuous if $\{f(z_n)\}_{n \in \mathbb{N}}$ is equicontinuous at s for every $s \in Y$.

PROPOSITION 3.1.15. *Let us assume the notations of Def. 3.1.13 and that \mathfrak{V} is full. Thus $\{v(x_\infty) \mid v \in \mathfrak{E}\} = \{\phi_1(x_\infty) \mid \phi \in \Phi\}$.*

PROOF. By definition follows the inclusion \subseteq . \mathfrak{V} being full we have $(\forall \phi \in \Phi)(\exists v \in \Gamma(\pi))(v(x_\infty) = \phi_1(x_\infty))$. Thus $(\forall \phi \in \Phi)(\exists v \in \mathcal{E})(v(x_\infty) = \phi_1(x_\infty))$ hence the inclusion \supseteq . \square

THEOREM 3.1.16 (MAIN1). *Let $\mathfrak{V} \doteq \langle \langle \mathfrak{E}, \tau \rangle, \pi, X, \|\cdot\| \rangle$ be a Banach bundle. Let $x_\infty \in X$ and $\mathcal{U}_0 \in \prod_{x \in X_0} \mathcal{C}(\mathbb{R}^+, B_s(\mathfrak{E}_x))$ be such that $\mathcal{U}_0(x)$ is a (C_0) -semigroup of contractions (respectively of isometries) on \mathfrak{E}_x for all $x \in X_0$.*

If $D(T_{x_\infty})$ is dense in \mathfrak{E}_{x_∞} and $\exists \lambda_0 > 0$ (respectively $\exists \lambda_0 > 0, \lambda_1 < 0$) such that the range $\mathcal{R}(\lambda_0 - T_{x_\infty})$ is dense in \mathfrak{E}_{x_∞} , (respectively the ranges $\mathcal{R}(\lambda_0 - T_{x_\infty})$ and $\mathcal{R}(\lambda_1 - T_{x_\infty})$ are dense in \mathfrak{E}_{x_∞}), then

$$\langle \mathcal{T}, x_\infty, \Phi \rangle \in \text{Graph}(\mathfrak{V}, \mathfrak{V}),$$

and T_{x_∞} in (3.1.2) is the generator of a C_0 -semigroup of contractions (respectively of isometries) on \mathfrak{E}_{x_∞} .

Moreover assume that $\{v(x) \mid v \in \mathcal{E}\}$ is dense in \mathfrak{E}_x for all $x \in X_0$, by taking the notations in (3.1.18), let $\mathfrak{W} \doteq \langle \langle \mathfrak{M}, \gamma \rangle, \rho, X, \mathfrak{R} \rangle$ and $\langle \mathfrak{V}, \mathfrak{W}, X, \mathbb{R}^+ \rangle$ be a (Θ, \mathcal{E}) -structure¹ such that (3.1.14) holds. Assume $\mathbf{U}_{\|\cdot\|_{B(\mathfrak{E}_z)}}(\mathcal{L}_{S_z}(\mathfrak{E}_z)) \subseteq \mathfrak{M}_z$ (respectively $\mathbf{U}_{is}(\mathcal{L}_{S_z}(\mathfrak{E}_z)) \subseteq \mathfrak{M}_z$) for all $z \in X$ ² and that there exists $F \in \Gamma(\rho)$ such that $F(x_\infty) = \mathcal{U}(x_\infty)$ and

- i: $\langle \mathfrak{V}, \mathfrak{W}, X, \mathbb{R}^+ \rangle$ has the $\mathbf{LD}_{x_\infty}(\{F\}, \mathcal{E})$; **or** it has the $\mathbf{LD}(\{F\}, \mathcal{E})$;
- ii: $(\forall v \in \mathcal{E})(\exists \phi \in \Phi)$ s.t. $\phi_1(x_\infty) = v(x_\infty)$ and $(\forall \{z_n\}_{n \in \mathbb{N}} \subset X \mid \lim_{n \in \mathbb{N}} z_n = x_\infty)$ we have that $\{\mathcal{U}(z_n)(\cdot)\phi_1(z_n) - F(z_n)(\cdot)v(z_n)\}_{n \in \mathbb{N}}$ is a bounded equicontinuous sequence;
- iii: X is metrizable.

Then $(\forall v \in \mathcal{E})(\forall K \in \text{Compact}(\mathbb{R}^+))$

$$(3.1.19) \quad \boxed{\lim_{z \rightarrow x_\infty} \sup_{s \in K} \|\mathcal{U}(z)(s)v(z) - F(z)(s)v(z)\| = 0,}$$

and

$$(3.1.20) \quad \boxed{\mathcal{U} \in \Gamma^{x_\infty}(\rho).}$$

¹ Well set indeed by Prop. 3.1.15, the density assumptions and Rem. 3.1.12 we have that S_x is dense in \mathfrak{E}_x for all $x \in X$.

² See Proposition 3.3.2 for models of \mathfrak{M} satisfying (3.1.14) and $\mathbf{U}_{\|\cdot\|_{B(\mathfrak{E}_z)}}(\mathcal{L}_{S_z}(\mathfrak{E}_z)) \subseteq \mathfrak{M}_z$.

In particular

$$(3.1.21) \quad \{\langle \mathcal{T}, x_\infty, \Phi \rangle\} \in \Delta_\Theta \langle \mathfrak{V}, \mathfrak{W}, \mathcal{E}, X, \mathbb{R}^+ \rangle.$$

Here \mathcal{T} and $D(T_{x_\infty})$ are defined as in Notations 3.1.1 with \mathcal{T}_0 and Φ given in (3.1.18), while $\mathcal{U} \in \prod_{x \in X} \mathfrak{M}_x$ such that $\mathcal{U} \upharpoonright X_0 \doteq \mathcal{U}_0$ and $\mathcal{U}(x_\infty)$ is the semigroup on \mathfrak{E}_{x_∞} generated by T_{x_∞} .

PROOF. By Lemma 3.1.3, [Kur, Lms.(2.8) – (2.9)], and the Hille-Yosida theorem, see [Kur, Th.(1.2)], we have the first sentence of the statement for the case of semigroup of contractions. By [BR, Corollary 3.1.19.] applied to T_x , for any $x \in X_0$, and by (3.1.7) we have $(\forall \lambda \in \mathbb{R})(\forall v_{x_\infty} \in \text{Dom}(T_{x_\infty}))$

$$(3.1.22) \quad \|(\mathbf{1} - \lambda T_{x_\infty})v_{x_\infty}\|_{x_\infty} \geq \|v_{x_\infty}\|_{x_\infty}.$$

Hence by [BR, Corollary 3.1.19.], T_{x_∞} will be a generator of a strongly continuous semigroup of isometries if we show that $\forall \lambda \in \mathbb{R} - \{0\}$

$$(3.1.23) \quad \mathcal{R}(\mathbf{1} - \lambda T_{x_\infty}) = \mathfrak{E}_{x_\infty}.$$

Let us set

$$\rho_0(T_{x_\infty}) \doteq \{\lambda \in \mathbb{R} - \{0\} \mid \mathcal{R}(\mathbf{1} - \lambda T) = \mathfrak{E}_x\}.$$

By (3.1.8) $\rho_0(T_{x_\infty}) = \rho(T_{x_\infty}) \cap (\mathbb{R} - \{0\})$, where $\rho(T_{x_\infty})$ is the resolvent set of T_x . By [DS, Lemma 7.3.2] $\rho(T_{x_\infty})$ is open in \mathbb{C} so $\rho_0(T_{x_\infty})$ is open in $\mathbb{R} - \{0\}$ with respect to the topology on $\mathbb{R} - \{0\}$ induced by that on \mathbb{C} . By Lemma 3.1.5 we deduce that $\rho_0(T_{x_\infty})$ is also closed in $\mathbb{R} - \{0\}$, therefore $\rho_0(T_{x_\infty}) = \mathbb{R} - \{0\}$ and (3.1.23) follows as well that T_{x_∞} is a generator of a strongly continuous semigroup of isometries.

Now we shall apply Lemma 2.2.5 in order to show the remaining part of the statement. By the Dupre' Thm., see for example [Kur, Cor. 2.10], and the fact that a metrizable space is completely regular, we deduce by hyp. (iii) that \mathfrak{V} is full. Let $v \in \mathcal{E}$ be fixed then by (3.1.18), $(\exists \phi \in \Phi)(v(x_\infty) = \phi_1(x_\infty))$ thus by (3.1.1) and Corollary 1.2.10

$$(3.1.24) \quad \lim_{z \rightarrow x_\infty} \|v(z) - \phi_1(z)\| = 0.$$

Now let $F \in \Gamma(\rho)$ of which in hypothesis so in particular

$$(3.1.25) \quad F(x_\infty) = \mathcal{U}(x_\infty),$$

moreover $\forall s \in \mathbb{R}^+$ and $z \in X$

$$\begin{aligned}
& \|\mathcal{U}(z)(s)v(z) - F(z)(s)v(z)\| \leq \\
& \|\mathcal{U}(z)(s)v(z) - \mathcal{U}(z)(s)\phi_1(z)\| + \|\mathcal{U}(z)(s)\phi_1(z) - F(z)(s)v(z)\| \leq \\
(3.1.26) \quad & \|v(z) - \phi_1(z)\| + \|\mathcal{U}(z)(s)\phi_1(z) - F(z)(s)v(z)\|.
\end{aligned}$$

For any $\lambda > 0$ let us set

$$g_\infty^\lambda \doteq (\lambda - T_{x_\infty})^{-1}\phi_1(x_\infty)$$

thus $g_\infty^\lambda \in \text{Dom}(T_{x_\infty})$ hence by Remark 3.1.7 and the construction of T_{x_∞} $\exists \psi^\lambda \in \Phi$ such that

$$(3.1.27) \quad \begin{cases} g_\infty^\lambda = \psi_1^\lambda(x_\infty) = \lim_{z \in x_\infty} \psi_1^\lambda(z) \\ T_{x_\infty} g_\infty^\lambda = \lim_{z \rightarrow x_\infty} T_z \psi_1^\lambda(z). \end{cases}$$

By (3.1.15) and (3.1.17) for all $z \in X$ and for all $w_z \in \bigcup_{v \in \mathcal{E}} v(z)$

$$(3.1.28) \quad \left(\int_0^\infty e^{-\lambda s} F(z)(s) ds \right) w_z = \int_0^\infty e^{-\lambda s} F(z)(s) w_z ds.$$

Moreover by the fact that \mathfrak{V} is full we have that for all $\phi \in \Phi$ there exists a $v \in \Gamma(\pi)$ such that $v(x_\infty) = \phi_1(x_\infty)$, thus by construction of \mathcal{E}

$$(3.1.29) \quad (\forall \phi \in \Phi)(\exists v \in \mathcal{E})(v(x_\infty) = \phi_1(x_\infty)).$$

Hence by (3.1.28), (3.1.29) and (3.1.25) for all $\phi \in \Phi$

$$(3.1.30) \quad \left(\int_0^\infty e^{-\lambda s} F(x_\infty)(s) ds \right) \phi_1(x_\infty) = \int_0^\infty e^{-\lambda s} \mathcal{U}(x_\infty)(s) \phi_1(x_\infty) ds.$$

Now set

$$\xi \doteq \mathfrak{L}(F),$$

thus by hypothesis (i) we have for all $\lambda > 0$

$$(3.1.31) \quad \xi(\cdot)(\lambda)v(\cdot) \in \Gamma^{x_\infty}(\pi).$$

Moreover

$$\begin{aligned}
(3.1.32) \quad \xi(x_\infty)(\lambda)v(x_\infty) &= \xi(x_\infty)(\lambda)\phi_1(x_\infty) \\
&= \int_0^\infty e^{-\lambda s} \mathcal{U}(x_\infty)(s)\phi_1(x_\infty) ds \text{ by (3.1.30)} \\
&= (\lambda - T_{x_\infty})^{-1}\phi_1(x_\infty) \text{ by [\textbf{Kur}, (1.3)]} \\
&\doteq g_\infty^\lambda = \psi_1^\lambda(x_\infty) \text{ by (3.1.27).}
\end{aligned}$$

By the fact that \mathfrak{V} is full, by (3.1.31), the fact that $\psi_1^\lambda \in \Gamma^{x_\infty}(\pi)$ by (3.1.1), by (3.1.32) and by Corollary 1.2.12 we have $\forall \lambda > 0$

$$(3.1.33) \quad \lim_{z \rightarrow x_\infty} \|\psi_1^\lambda(z) - \xi(z)(\lambda)v(z)\| = 0.$$

Now $(\forall \lambda > 0)(\forall z \in X)$ set

$$w^\lambda(z) \doteq (\lambda \mathbf{1} - T_z)\psi_1^\lambda(z),$$

thus

$$\begin{aligned}
(3.1.34) \quad & \left\| \int_0^\infty e^{-\lambda s} (\mathcal{U}(z)(s)\phi_1(z) - F(z)(s)v(z)) ds \right\| \leq \\
& \left\| \int_0^\infty e^{-\lambda s} \mathcal{U}(z)(s)(\phi_1(z) - w^\lambda(z)) ds \right\| + \left\| \int_0^\infty e^{-\lambda s} (\mathcal{U}(z)(s)w^\lambda(z) - F(z)(s)v(z)) ds \right\| \leq \\
& \frac{1}{\lambda} \|\phi_1(z) - w^\lambda(z)\| + \|\psi_1^\lambda(z) - \xi(z)(\lambda)v(z)\|.
\end{aligned}$$

Here we consider that by hypothesis and by the first part of the statemet $\|\mathcal{U}(z)\| \leq 1$ for all $z \in X$, moreover we applied the Hille-Yosida formula [\textbf{Kur}, (1.3)]. Now

$$\begin{aligned}
(3.1.35) \quad & \|\phi_1(z) - w^\lambda(z)\| = \\
& \|\phi_1(z) - (\lambda \mathbf{1} - T_z)\psi_1^\lambda(z)\| \leq \\
& \|\phi_1(z) - v(z)\| + \|v(z) - \lambda \xi(z)(\lambda)v(z) + \lambda \xi(z)(\lambda)v(z) - (\lambda \mathbf{1} - T_z)\psi_1^\lambda(z)\| \leq \\
& \|\phi_1(z) - v(z)\| + \lambda \|\xi(z)(\lambda)v(z) - \psi_1^\lambda(z)\| + \|T_z \psi_1^\lambda(z) - (\lambda \xi(z)(\lambda)v(z) - v(z))\|.
\end{aligned}$$

By (3.1.27) $T_{x_\infty} \psi_1^\lambda(x_\infty) = T_{x_\infty} g_\infty^\lambda$ moreover

$$\begin{aligned}
(3.1.36) \quad & T_{x_\infty} g_\infty^\lambda = -(\lambda - T_{x_\infty})g_\infty^\lambda + \lambda g_\infty^\lambda \\
& = -(\lambda - T_{x_\infty})(\lambda - T_{x_\infty})^{-1}\phi_1(x_\infty) + \lambda g_\infty^\lambda \\
& = \lambda g_\infty^\lambda - \phi_1(x_\infty) = \lambda \xi(x_\infty)(\lambda)v(x_\infty) - v(x_\infty),
\end{aligned}$$

where in the last equality we used (3.1.32) and the construction of ϕ . By (3.1.27) we have that $(X \ni z \mapsto T_z \psi_1^\lambda(z)) \in \Gamma^{x_\infty}(\pi)$, hence by (3.1.36), the fact that $\lambda\xi(\cdot)(\lambda)v(\cdot) - v \in \Gamma^{x_\infty}(\pi)$ by (3.1.31), we deduce by the fact that \mathfrak{V} is full and by Corollary 1.2.12 that $\forall \lambda > 0$

$$(3.1.37) \quad \lim_{z \rightarrow x_\infty} \|T_z \psi_1^\lambda(z) - (\lambda\xi(z)(\lambda)v(z) - v(z))\| = 0.$$

Therefore by (3.1.35), (3.1.24), (3.1.33) and (3.1.37)

$$\lim_{z \rightarrow x_\infty} \|\phi_1(z) - w^\lambda(z)\| = 0.$$

By this one along with (3.1.33) we can state by using (3.1.34) that $\forall \lambda > 0$

$$\lim_{z \rightarrow x_\infty} \left\| \int_0^\infty e^{-\lambda s} (\mathcal{U}(z)(s)\phi_1(z) - F(z)(s)v(z)) ds \right\| = 0.$$

Therefore $\forall \lambda > 0$ and $(\forall \{z_n\}_{n \in \mathbb{N}} \subset X \mid \lim_{n \in \mathbb{N}} z_n = x_\infty)$

$$(3.1.38) \quad \lim_{n \in \mathbb{N}} \left\| \int_0^\infty e^{-\lambda s} (\mathcal{U}(z_n)(s)\phi_1(z_n) - F(z_n)(s)v(z_n)) ds \right\| = 0.$$

By (3.1.38), hypothesis (ii) and [Kur, Lemma (2.11)] we have $(\forall \{z_n\}_{n \in \mathbb{N}} \subset X \mid \lim_{n \in \mathbb{N}} z_n = x_\infty)$ and $\forall K \in Compact(\mathbb{R}^+)$

$$\limsup_{n \in \mathbb{N}} \sup_{s \in K} \|\mathcal{U}(z_n)(s)\phi_1(z_n) - F(z_n)(s)v(z_n)\| = 0.$$

Therefore by hypothesis (iii) and [GT, Prop.10, §2.6, Ch. 9], $\forall K \in Compact(\mathbb{R}^+)$

$$(3.1.39) \quad \lim_{z \rightarrow x_\infty} \sup_{s \in K} \|\mathcal{U}(z)(s)\phi_1(z) - F(z)(s)v(z)\| = 0.$$

In conclusion by (3.1.39), (3.1.24) and (3.1.26) we obtain $\forall K \in Compact(\mathbb{R}^+)$

$$(3.1.40) \quad \lim_{z \rightarrow x_\infty} \sup_{s \in K} \|\mathcal{U}(z)(s)v(z) - F(z)(s)v(z)\| = 0,$$

hence (3.1.19). By (3.1.17) and (3.1.40) we obtain (2.2.6). Thus (3.1.20) and (3.1.21) follow by Lemma 2.2.5, by (3.1.17) and by the following one $\forall K \in Compact(\mathbb{R}^+)$ and $\forall v \in \mathcal{E}$

$$\sup_{z \in X} \sup_{s \in K} \|\mathcal{U}(z)(s)v(z)\| \leq \sup_{z \in X} \|v(z)\| < \infty.$$

where we considered that by construction $\|\mathcal{U}(z)(s)\| \leq 1$, for all $s \in \mathbb{R}^+$ and $z \in X$ and that $v \in \Gamma(\pi)$. \square

REMARK 3.1.17. *If \mathfrak{W} is full $(\exists F \in \Gamma(\rho))(F(x_\infty) = \mathcal{U}(x_\infty))$, so hypotheses reduce.*

3.2. Corollary I. Constructions of Equicontinuous sequence

As the first corollary we give conditions in order to satisfy the bounded equicontinuity of which in hypothesis (ii).

COROLLARY 3.2.1. *Let us assume the hypotheses of Theorem 3.1.16 except (ii) replaced by the following one*

$$(\exists G \in \prod_{z \in X} \mathfrak{L}_1(\mathbb{R}^+, \mathcal{L}_{S_x}(\mathfrak{E}_x)) (\exists H \in \prod_{z \in X}^b \mathcal{L}(\mathfrak{E}_z)) (\exists F \in \Gamma(\rho))$$

such that $F(x_\infty) = \mathcal{U}(x_\infty)$ and $\forall s > 0$

$$(3.2.1) \quad \begin{cases} \sup_{x \in X} \sup_{s > 0} \|F(x)(s)\| < \infty \\ (\forall s_1 > 0)(\exists a > 0)(\sup_{u \in [s_1, s]} \sup_{z \in X} \|G(z)(u)\| \leq a|s - s_1|) \\ (\forall z \in X)(F(z)(s) = H(z) + \int_0^s G(z)(u) du), \end{cases}$$

where the integration is with respect to the Lebesgue measure on $[0, s]$ and with respect to the lct on $\mathcal{L}_{S_z}(\mathfrak{E}_z)$. Then holds the statement of Theorem 3.1.16.

PROOF. Let $v \in \mathcal{E}$ thus $(\exists \phi \in \Phi)(v(x_\infty) = \phi_1(x_\infty))$ so $(\forall \{z_n\}_{n \in \mathbb{N}} \subset X \mid \lim_{n \in \mathbb{N}} z_n = x_\infty)$ we have

$$\sup_{n \in \mathbb{N}} \sup_{s > 0} \|\mathcal{U}(z_n)(s)\phi_1(z_n) - F(z_n)(s)v(z_n)\| \leq \sup_{n \in \mathbb{N}} \|\phi_1(z_n)\| + M \sup_{n \in \mathbb{N}} \|v(z_n)\| < \infty.$$

Here in the first inequality we used $\|\mathcal{U}(z)(s)\| \leq 1$ for all $z \in X$ and $s > 0$ by construction, and $M \doteq \sup_{z \in X} \sup_{s > 0} \|F(z)(s)\| < \infty$ by hypothesis, while in the second inequality we used the fact that $v \in \prod_{x \in X}^b \mathfrak{E}_x$, by construction and that $\sup_{n \in \mathbb{N}} \|\phi_1(z_n)\| < \infty$ because of $\exists \overline{\lim}_{n \in \mathbb{N}} \|\phi_1(z_n)\| \in \mathbb{R}$ by Remark 3.1.7 and by construction $\|\cdot\|$ is *u.s.c.* Moreover by [Kur, (1.4)], (3.2.1) and $S_x = \{\{w(x)\} \mid w \in \mathcal{E}\}$ for all $x \in X$ we have

$$\begin{aligned} \mathcal{U}(z_n)(s)\phi_1(z_n) - F(z_n)(s)v(z_n) &= \int_0^s (\mathcal{U}(z_n)(u)T_{z_n}\phi_1(z_n) - G(z_n)(u)v(z_n)) du + \\ &\quad + \phi_1(z_n) - H(z_n)v(z_n). \end{aligned}$$

Thus for any $s_1, s_2 \in \mathbb{R}^+$

$$\begin{aligned} \sup_{n \in \mathbb{N}} \|(\mathcal{U}(z_n)(s_1)\phi_1(z_n) - F(z_n)(s_1)v(z_n)) - (\mathcal{U}(z_n)(s_2)\phi_1(z_n) - F(z_n)(s_2)v(z_n))\| \leq \\ |s_1 - s_2| \sup_{n \in \mathbb{N}} \sup_{u \in [s_1, s_2]} \|\mathcal{U}(z_n)(u)T_{z_n}\phi_1(z_n) - G(z_n)(u)v(z_n)\| \leq \\ |s_1 - s_2| \sup_{n \in \mathbb{N}} (\|T_{z_n}\phi_1(z_n)\| - a\|v(z_n)\|) \leq J|s_1 - s_2|. \end{aligned}$$

Here in the second inequality we used $\|\mathcal{U}(z)(u)\| \leq 1$ by construction and the hypothesis, in the third one the fact that $\sup_{n \in \mathbb{N}} \|T_{z_n}\phi_1(z_n)\| < \infty$ as well $\sup_{n \in \mathbb{N}} \|v(z_n)\| < \infty$, because of $\exists \overline{\lim}_{n \in \mathbb{N}} \|T_{z_n}\phi_1(z_n)\| \in \mathbb{R}$ and $\exists \overline{\lim}_{n \in \mathbb{N}} \|v(z_n)\| \in \mathbb{R}$ due to the fact that $\|\cdot\|$ is *u.s.c.* by construction and Remark 3.1.7 for the first limit and the continuity of v for the second one. Therefore hypothesis (ii) of Theorem 3.1.16 is satisfied, hence the statement follows by Theorem 3.1.16. \square

3.3. Corollaries II. Construction of $\langle \mathfrak{V}, \mathfrak{W}, X, \mathbb{R}^+ \rangle$ with the LD

Let us start with the following simple result about the relation among full and Laplace duality property.

PROPOSITION 3.3.1. *Let $\mathfrak{W} \doteq \langle \langle \mathfrak{M}, \gamma \rangle, \rho, X, \mathfrak{R} \rangle$ and $\langle \mathfrak{V}, \mathfrak{W}, X, \mathbb{R}^+ \rangle$ be a (Θ, \mathcal{E}) -structure such that \mathfrak{V} is a Banach bundle and $x_\infty \in X$. Assume that*

- (1) \mathfrak{V} and \mathfrak{W} are full;
- (2) $\mathcal{E} = \Gamma(\pi)$ and Θ is given in (3.1.16);
- (3) $(\forall F \in \Gamma^{x_\infty}(\rho))(M(F) \doteq \sup_{x \in X} \sup_{s \in \mathbb{R}^+} \|F(x)(s)\| < \infty)$;
- (4) $(\forall \sigma \in \Gamma(\rho))(\sup_{x \in X} \sup_{s \in \mathbb{R}^+} \|\sigma(x)(s)\| < \infty)$;
- (5) X is metrizable.

If $\langle \mathfrak{V}, \mathfrak{W}, X, \mathbb{R}^+ \rangle$ has the LD then it has the \mathbf{LD}_{x_∞} .

PROOF. Let $F \in \Gamma^{x_\infty}(\rho)$ and $w \in \Gamma^{x_\infty}(\pi)$ thus by hypothesis (2) and Corollary 1.2.10 there exist $\sigma \in \Gamma(\rho)$ and $v \in \Gamma(\pi)$ such that $\sigma(x_\infty) = F(x_\infty)$, $v(x_\infty) = w(x_\infty)$, and $\forall K \in \text{Comp}(\mathbb{R}^+)$, $\forall v \in \mathcal{E}$

$$(3.3.1) \quad \begin{cases} \lim_{z \rightarrow x_\infty} \|w(z) - v(z)\| = 0 \\ \lim_{z \rightarrow x_\infty} \sup_{s \in K} \|(F(x)(s) - \sigma(x)(s))v(x)\| = 0. \end{cases}$$

Moreover $\forall \lambda > 0$

$$\begin{aligned}
& \left\| \int_0^\infty e^{\lambda s} F(z)(s) w(z) ds - \int_0^\infty e^{\lambda s} \sigma(z)(s) v(z) ds \right\| \leq \\
& \left\| \int_0^\infty e^{\lambda s} F(z)(s) (w(z) - v(z)) ds \right\| + \left\| \int_0^\infty e^{\lambda s} (F(z)(s) - \sigma(z)(s)) v(z) ds \right\| \leq \\
(3.3.2) \quad & \frac{1}{\lambda} M(F) \|v(z) - w(z)\| + \int_0^\infty e^{\lambda s} \|(F(z)(s) - \sigma(z)(s)) v(z)\| ds.
\end{aligned}$$

By the hypotheses (3 – 4) $\sup_{z \in X} \sup_{s \in \mathbb{R}^+} \|(F(z)(s) - \sigma(z)(s)) v(z)\| < \infty$ hence $\forall \{z_n\}_{n \in \mathbb{N}} \subset X$ such that $\lim_{n \in \mathbb{N}} z_n = x_\infty$ we have by (3.3.1), (3.3.2) and a well-known theorem on convergence of sequences of integrals that $\forall \lambda > 0$

$$\lim_{n \in \mathbb{N}} \left\| \int_0^\infty e^{\lambda s} F(z_n)(s) w(z_n) ds - \int_0^\infty e^{\lambda s} \sigma(z_n)(s) v(z_n) ds \right\| = 0.$$

Thus $\forall \lambda > 0$ by hypothesis (5)

$$\lim_{z \rightarrow x_\infty} \left\| \int_0^\infty e^{\lambda s} F(z)(s) w(z) ds - \int_0^\infty e^{\lambda s} \sigma(z)(s) v(z) ds \right\| = 0,$$

hence the statement by Corollary 1.2.10. \square

Now we shall see that in the case of a bundle of normed space we can choose for all x a simple space \mathfrak{M}_x satisfying (3.1.14).

PROPOSITION 3.3.2. *Let $\mathfrak{W} \doteq \langle \langle \mathfrak{M}, \gamma \rangle, \rho, X, \mathfrak{R} \rangle$ and $\langle \mathfrak{V}, \mathfrak{W}, X, \mathbb{R}^+ \rangle$ be a (Θ, \mathcal{E}) –structure such that for all $x \in X$, \mathfrak{E}_x is a reflexive Banach space, $S_x \subseteq \mathcal{P}_\omega(\mathfrak{E}_x)$ and*

$$\mathfrak{M}_x \subseteq \left\{ F \in \mathcal{C}_c(\mathbb{R}^+, \mathcal{L}_{S_x}(\mathfrak{E}_x)) \mid (\forall \lambda > 0) \left(\int_{\mathbb{R}^+}^* e^{-\lambda s} \|F(s)\|_{B(\mathfrak{E}_x)} ds < \infty \right) \right\}.$$

Thus

$$(3.3.3) \quad \mathfrak{M}_x \subset \bigcap_{\lambda > 0} \mathfrak{L}_1(\mathbb{R}^+, \mathcal{L}_{S_x}(\mathfrak{E}_x); \mu_\lambda).$$

In particular (3.3.3), and $\mathbf{U}_{\|\cdot\|_{B(\mathfrak{E}_x)}}(\mathcal{L}_{S_x}(\mathfrak{E}_x)) \subseteq \mathfrak{M}_x$ hold if for any $x \in X$

$$\mathfrak{M}_x = \left\{ F \in \mathcal{C}_c(\mathbb{R}^+, \mathcal{L}_{S_x}(\mathfrak{E}_x)) \mid \sup_{s \in \mathbb{R}^+} \|F(s)\|_{B(\mathfrak{E}_x)} < \infty \right\}.$$

PROOF. The first sentence follows by [Sil, Corollary 2.6.], while the second sentence comes by the first one. \square

3.3.1. U–Spaces.

NOTATIONS 3.3.3. First of all recall that for any W, Z topological vector spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ we denote by $\mathcal{L}(W, Z)$ the \mathbb{K} –linear space of all continuous linear map on W and with values in Z and set $\mathcal{L}(Z) \doteq \mathcal{L}(Z, Z)$ and $Z^* \doteq \mathcal{L}(Z, \mathbb{K})$. In this section we assume fixed the following data:

- (1) a set X , a locally compact space Y and a Radon measure μ on Y ;
- (2) a family $\{\mathfrak{E}_x\}_{x \in X}$ of Hlcs;
- (3) a family $\{\tau_x\}_{x \in X}$ such that $\langle \mathcal{L}(\mathfrak{E}_x), \tau_x \rangle \in \text{Hlcs}$, $\forall x \in X$;
- (4) a family $\{\mathfrak{N}_x\}_{x \in X}$ such that $\mathfrak{N}_x \doteq \{\nu_{j_x}^x \mid j_x \in J_x\}$ is a fundamental set of seminorms on \mathfrak{E}_x , $\forall x \in X$;
- (5) a family $\{Q_x\}_{x \in X}$ such that $Q_x \doteq \{q_{\alpha_x}^x \mid \alpha_x \in A_x\}$ is a fundamental set of seminorms on $\langle \mathcal{L}(\mathfrak{E}_x), \tau_x \rangle$, $\forall x \in X$;
- (6) $\langle \mathcal{H}, \mathfrak{T} \rangle \in \text{Hlcs}$ such that
 - $\mathcal{H} \subseteq \prod_{x \in X} \mathfrak{E}_x$ as linear spaces;
 - $\iota_x(\mathfrak{E}_x) \subset \mathcal{H}$, for all $x \in X$;
 - $\text{Pr}_x \in \mathcal{L}(\langle \mathcal{H}, \mathfrak{T} \rangle, \mathfrak{E}_x)$ and $\iota_x \in \mathcal{L}(\mathfrak{E}_x, \langle \mathcal{H}, \mathfrak{T} \rangle)$, for all $x \in X$;
 - $\exists \mathcal{A} \subseteq \prod_{x \in X} \mathcal{L}(\mathfrak{E}_x)$ linear space such that
 - (a) $\theta(\mathcal{A}) \upharpoonright \mathcal{H} \subseteq \mathcal{L}(\langle \mathcal{H}, \mathfrak{T} \rangle)$,
 - (b) $\iota_x(\mathcal{L}(\mathfrak{E}_x)) \subseteq \mathcal{A}$ for all $x \in X$.

Here θ is defined in Def. 3.3.9.

For any $Z \in \text{Hlcs}$ we denote by $\mathfrak{L}_1(Y, Z, \mu)$ the linear space of all maps on Y and with values in Z which are essentially μ –integrable in the sense described in [INT, Ch. 6]. While $\mathcal{C}_{cs}(Y, Z)$ denotes the linear space of all continuous maps $f : Y \rightarrow Z$ with compact support. Moreover for any family $\{Z_x\}_{x \in X}$ of linear spaces and for all $x \in X$ set $\text{Pr}_x : \prod_{y \in X} Z_y \ni f \mapsto f(x) \in Z_x$ and $\iota_x : Z_x \rightarrow \prod_{y \in X} Z_y$ such that for all $x \neq y$ and $z_x \in Z_x$ $\text{Pr}_y \circ \iota_x(z_x) = \mathbf{0}_y$, while $\text{Pr}_x \circ \iota_x = \text{Id}_x$.

Finally we set

$$\langle \cdot, \cdot \rangle : \text{End}(\mathcal{H}) \times \mathcal{H} \ni (A, v) \mapsto A(v) \in \mathcal{H},$$

and for all $x \in X$

$$\langle \cdot, \cdot \rangle_x : \text{End}(\mathfrak{E}_x) \times \mathfrak{E}_x \ni (A, v) \mapsto A(v) \in \mathfrak{E}_x.$$

DEFINITION 3.3.4. Let $\mathfrak{W} \doteq \langle \langle \mathfrak{M}, \gamma \rangle, \rho, X, \mathfrak{R} \rangle$ be a bundle of Ω –spaces such that for all $x \in X$

$$\mathfrak{M}_x \subseteq \mathfrak{L}_1(Y, \langle \mathcal{L}(\mathfrak{E}_x), \tau_x \rangle; \mu).$$

Set

$$(3.3.4) \quad \begin{cases} \blacksquare_{\mu} : \prod_{x \in X} \mathfrak{L}_1(Y, \langle \mathcal{L}(\mathfrak{E}_x), \tau_x \rangle; \mu) \times \prod_{x \in X} \mathfrak{E}_x \rightarrow \prod_{x \in X} \mathfrak{E}_x \\ \blacksquare_{\mu}(H, v)(x) \doteq \left\langle \int_{\mathbb{R}^+} H(x)(s) d\mu(s), v(x) \right\rangle_x \in \mathfrak{E}_x. \end{cases}$$

REMARK 3.3.5. Let $\langle \mathfrak{V}, \mathfrak{W}, X, \mathbb{R}^+ \rangle$ be a (Θ, \mathcal{E}) -structure satisfying (3.1.14) and $\mathcal{O} \subseteq \Gamma(\rho)$, $\mathcal{D} \subseteq \Gamma(\pi)$. Then

$$(3.3.5) \quad \mathbf{LD}(\mathcal{O}, \mathcal{D}) \Leftrightarrow (\forall \lambda > 0) (\blacksquare_{\mu_{\lambda}}(\mathcal{O}, \mathcal{D}) \subseteq \Gamma(\pi)).$$

Similarly for all $x \in X$

$$(3.3.6) \quad \mathbf{LD}_x(\mathcal{O}, \mathcal{D}) \Leftrightarrow (\forall \lambda > 0) (\blacksquare_{\mu_{\lambda}}(\Gamma_{\mathcal{O}}^x(\rho), \Gamma_{\mathcal{D}}^x(\pi)) \subseteq \Gamma^x(\pi)).$$

DEFINITION 3.3.6 (U-Spaces). \mathfrak{G} is a U-space with respect to $\{\langle \mathcal{L}(\mathfrak{E}_x), \tau_x \rangle\}_{x \in X}$, \mathfrak{T} and D iff

- (1) $\mathfrak{G} \in Hlcs$;
- (2) $\mathfrak{G} \subset \mathcal{L}(\langle \mathcal{H}, \mathfrak{T} \rangle)$ as linear spaces;
- (3) $D \subseteq \mathcal{H}$;
- (4) $(\forall T \in lcp) (\exists \Psi_T \in \text{End} [\text{End}(\mathcal{H})^T, \prod_{x \in X} \text{End}(\mathfrak{E}_x)^Y]) (\forall \nu \in \text{Radon}(T))$

$$\Psi_T : \mathfrak{L}_1(T, \mathfrak{G}, \nu) \rightarrow \prod_{x \in X} \mathfrak{L}_1(T, \langle \mathcal{L}(\mathfrak{E}_x), \tau_x \rangle; \nu)^3$$

$$\text{and } \forall \overline{F} \in \mathfrak{L}_1(T, \mathfrak{G}, \nu), \forall v \in D. \forall x \in X$$

$$(3.3.7) \quad \left\langle \int \Psi_T(\overline{F})(x)(s) d\nu(s), v(x) \right\rangle_x = \left\langle \int \overline{F}(s) d\nu(s), v \right\rangle(x)$$

The reason of introducing the concept of U-spaces will be clarified by the following

PROPOSITION 3.3.7. Let $\langle \mathfrak{V}, \mathfrak{W}, X, \mathbb{R}^+ \rangle$ be a (Θ, \mathcal{E}) -structure satisfying (3.1.14), and let \mathfrak{G} be a U-space with respect to $\{\mathcal{L}_{S_x}(\mathfrak{E}_x)\}_{x \in X}$, \mathfrak{T} and \mathcal{D} . Then $\forall \lambda > 0$, $\overline{F} \in \mathfrak{L}_1(\mathbb{R}^+, \mathfrak{G}, \mu_{\lambda})$, $v \in \mathcal{D}$

$$(3.3.8) \quad \blacksquare_{\mu_{\lambda}}(\Psi_{\mathbb{R}^+}(\overline{F}), v) = \left\langle \int \overline{F}(s) d\mu_{\lambda}(s), v \right\rangle.$$

Moreover if $\exists \mathcal{F} \subset \bigcap_{\lambda > 0} \mathfrak{L}_1(\mathbb{R}^+, \mathfrak{G}, \mu_{\lambda})$ such that $\Psi_{\mathbb{R}^+}(\mathcal{F}) = \mathcal{O}$ then

$$(3.3.9) \quad \boxed{\mathbf{LD}(\mathcal{O}, \mathcal{D}) \Leftrightarrow (\forall \lambda > 0) (\langle \mathcal{B}_{\lambda}, \mathcal{D} \rangle \subseteq \Gamma(\pi)).}$$

³ Of course Ψ_T here has to be understood as $\Psi_T \upharpoonright \mathfrak{L}_1(T, \mathfrak{G}, \nu)$.

Here

$$\mathcal{B}_\lambda \doteq \left\{ \int \overline{F}(s) d\mu_\lambda(s) \mid \overline{F} \in \mathcal{F} \right\}.$$

REMARK 3.3.8. In particular if $\exists \mathcal{F} \subset \bigcap_{\lambda>0} \mathcal{L}_1(\mathbb{R}^+, \mathfrak{G}, \mu_\lambda)$ such that $\Psi_{\mathbb{R}^+}(\mathcal{F}) = \mathcal{O}$ then

$$\langle \mathfrak{G}, \mathcal{D} \rangle \subseteq \Gamma(\pi) \Rightarrow \mathbf{LD}(\mathcal{O}, \mathcal{D}).$$

More in general if $\exists \mathfrak{G}_0$ complete subspace of \mathfrak{G} and $\exists \mathcal{F} \subset \{\overline{F} \in \bigcap_{\lambda>0} \mathcal{L}_1(\mathbb{R}^+, \mathfrak{G}, \mu_\lambda) \mid \overline{F}(\mathbb{R}^+) \subseteq \mathfrak{G}_0\}$ such that $\Psi_{\mathbb{R}^+}(\mathcal{F}) = \mathcal{O}$ then

$$\langle \mathfrak{G}_0, \mathcal{D} \rangle \subseteq \Gamma(\pi) \Rightarrow \mathbf{LD}(\mathcal{O}, \mathcal{D}).$$

PROOF. (3.3.8) follows by (3.3.7), while (3.3.9) follows by (3.3.8) and Remark 3.3.5. \square

Thus the **U** property expressed by (3.3.7) is an important tool for ensuring the satisfaction of the **LD**. For this reason the remaining of the present section will be dedicated to the construction of a space \mathfrak{G} , Def. (3.3.12), which is a **U**–space, see Theorem 3.3.27 and Corollary 3.3.30 for the $\mathbf{LD}(\mathcal{O}, \mathcal{D})$.

DEFINITION 3.3.9. Set

$$\begin{cases} \chi_{\mathcal{H}} : \text{End}(\mathcal{H}) \rightarrow \prod_{x \in X} \text{End}(\mathfrak{E}_x), \\ (\forall x \in X)(\forall w \in \text{End}(\mathcal{H}))((\text{Pr}_x \circ \chi_{\mathcal{H}})(w) = \text{Pr}_x \circ w \circ \iota_x), \\ \chi \doteq \chi_{\prod_{x \in X} \mathfrak{E}_x}. \end{cases}$$

Well defined indeed by construction $\iota_x(\mathfrak{E}_x) \subset \mathcal{H}$, for all $x \in X$. Finally set

$$\begin{cases} \theta : \prod_{x \in X} \text{End}(\mathfrak{E}_x) \rightarrow \text{End}(\prod_{x \in X} \mathfrak{E}_x), \\ (\forall x \in X)(\forall u \in \prod_{x \in X} \text{End}(\mathfrak{E}_x))(\text{Pr}_x \circ \theta(u) = \text{Pr}_x(u) \circ \text{Pr}_x), \\ \theta_{\mathcal{H}} : \text{Im}(\chi_{\mathcal{H}}) \ni u \mapsto \theta(u) \upharpoonright \mathcal{H}. \end{cases}$$

Well-posed by applying [**ALG**, Prop. 4, n°5, §1, Ch. 2].

REMARK 3.3.10. $(\forall x \in X)(\forall u \in \prod_{x \in X} \text{End}(\mathfrak{E}_x))$ we have $(\text{Pr}_x \circ \theta(u) \circ \iota_x = \text{Pr}_x(u))$.

PROPOSITION 3.3.11. The space $\prod_{x \in X} \mathfrak{E}_x$ with the product topology satisfies the request for the space $\langle \mathcal{H}, \mathfrak{T} \rangle$ in Notations 3.3.3 with the choice $\mathcal{A} = \prod_{x \in X} \mathcal{L}(\mathfrak{E}_x)$.

PROOF. $\text{Pr}_x \in \mathcal{L}(\prod_{y \in X} \mathfrak{E}_y, \mathfrak{E}_x)$ by definition of the product topology, moreover $\iota_x \in \mathcal{L}(\mathfrak{E}_x, \prod_{y \in X} \mathfrak{E}_y)$. Indeed ι_x is clearly linear and by considering that for any

net $\{f^\alpha\}_{\alpha \in D}$ and any f in $\prod_{y \in X} \mathfrak{E}_y$, $\lim_{\alpha \in D} f^\alpha = f$ if and only if $\lim_{\alpha \in D} f^\alpha(y) = f(y)$ for all $y \in X$, we deduce that for any net $\{f_x^\alpha\}_{\alpha \in D}$ and any f_x in \mathfrak{E}_x such that $\lim_{\alpha \in D} f_x^\alpha = f_x$ we have $\lim_{\alpha \in D} \iota_x(f_x^\alpha) = \iota_x(f_x)$, so ι_x is continuous. Let $x \in X$ and $u \in \prod_{x \in X} \mathcal{L}(\mathfrak{E}_x)$ so $\text{Pr}_x(u) \circ \text{Pr}_x \in \mathcal{L}(\prod_{y \in X} \mathfrak{E}_y, \mathfrak{E}_x)$, so (6a) follows by the definition of θ and [GT, Prp. 4, No3, §2]. Finally (6b) is trivial. \square

The following is the main structure of the present section. For the definition and properties of locally convex final topologies see [TVS, No4, §4].

DEFINITION 3.3.12. *Set for all $x \in X$*

$$\begin{cases} G \doteq \theta(\mathcal{A}) \upharpoonright \mathcal{H}, \\ g_x : \mathcal{L}(\mathfrak{E}_x) \ni f_x \mapsto \iota_x \circ f_x \circ \text{Pr}_x \in \text{End}(\prod_{y \in X} \mathfrak{E}_y) \\ h_x : \mathcal{L}(\mathfrak{E}_x) \ni f_x \mapsto g_x(f_x) \upharpoonright \mathcal{H}. \end{cases}$$

We shall denote by \mathfrak{G} the lcs G provided with the locally convex final topology of the family of topologies $\{\tau_x\}_{x \in X}$ of the $\{\mathcal{L}(\mathfrak{E}_x)\}_{x \in X}$, for the family of linear mappings $\{h_x\}_{x \in X}$.

DEFINITION 3.3.13. *Set in $\prod_{x \in X} \text{End}(\mathfrak{E}_x)$ the following binary operation \circ . For all $x \in X$ we set $\text{Pr}_x(f \circ h) \doteq f(x) \circ h(x)$.*

It is easy to verify that $\langle \prod_{x \in X} \text{End}(\mathfrak{E}_x), +, \circ \rangle$ is an algebra over \mathbb{K} as well as $\langle \prod_{x \in X} \mathcal{L}(\mathfrak{E}_x), +, \circ \rangle$.

LEMMA 3.3.14. *$G \subset \mathcal{L}(\langle \mathcal{H}, \mathfrak{T} \rangle)$, moreover θ is a morphism of algebras. Finally if \mathcal{A} is a subalgebra of $\prod_{x \in X} \mathcal{L}(\mathfrak{E}_x)$ then G is a subalgebra of $\mathcal{L}(\langle \mathcal{H}, \mathfrak{T} \rangle)$.*

PROOF. The first sentence is immediate by (6a) in Notations 3.3.3. Let $u, v \in \prod_{x \in X} \mathcal{L}(\mathfrak{E}_x)$ thus for all $x \in X$

$$\begin{aligned} \text{Pr}_x \circ \theta(u \circ v) &= (u(x) \circ v(x)) \circ \text{Pr}_x \\ &= u(x) \circ \text{Pr}_x \circ \theta(v) \\ &= \text{Pr}_x \circ \theta(u) \circ \theta(v), \end{aligned}$$

so $\theta(u \circ v) = \theta(u) \circ \theta(v)$, similarly we can show that θ is linear by the linearity of Pr_x for all $x \in X$. Thus θ is a morphism of algebras, so the last sentence of the statement follows by the first one and the fact that \mathcal{A} is an algebra. \square

PROPOSITION 3.3.15. $\theta_{\mathcal{H}} \circ \chi_{\mathcal{H}}(w) = w \circ \iota_x \circ \text{Pr}_x \upharpoonright \mathcal{H}$ for all $w \in \text{End}(\mathcal{H})$, Moreover $\theta_{\mathcal{H}}(\text{Im}(\chi_{\mathcal{H}})) \subset \text{Dom}(\chi_{\mathcal{H}})$ and $\chi_{\mathcal{H}} \circ \theta_{\mathcal{H}} = \text{Id} \upharpoonright \text{Im}(\chi_{\mathcal{H}})$.

PROOF. Let $w \in \text{End}(\mathcal{H})$ thus for all $x \in X$ we have $(\text{Pr}_x \circ \theta_{\mathcal{H}} \circ \chi_{\mathcal{H}})(w) = \text{Pr}_x(\chi_{\mathcal{H}}(w)) \circ \text{Pr}_x \upharpoonright \mathcal{H} = \text{Pr}_x \circ w \circ \iota_x \circ \text{Pr}_x \upharpoonright \mathcal{H}$ and the first sentence of the statement follows. By the first sentence and the assumption that $\iota_x(\mathfrak{E}_x) \subset \mathcal{H}$ we have $\theta(\text{Im}(\chi_{\mathcal{H}})) \upharpoonright \mathcal{H} \subset \text{End}(\mathcal{H})$ so $\chi_{\mathcal{H}} \circ \theta_{\mathcal{H}}$ is well set. Moreover for all $x \in X$ and $u \in \text{Im}(\chi_{\mathcal{H}})$ we have $\text{Pr}_x(\chi_{\mathcal{H}}(\theta(u) \upharpoonright \mathcal{H})) = \text{Pr}_x \circ \theta(u) \circ \iota_x = \text{Pr}_x(u) \circ \text{Pr}_x \circ \iota_x = \text{Pr}_x(u)$. \square

PROPOSITION 3.3.16. Let $x \in X$, then

- (1) $g_x = \theta \circ \iota_x$ so $\text{Im}(h_x) \subseteq G$;
- (2) $h_x \in \text{End}(\mathcal{L}(\mathfrak{E}_x), G)$;
- (3) $\exists h_x^{-1} : \text{Im}(h_x) \rightarrow \mathcal{L}(\mathfrak{E}_x)$ and

$$\begin{cases} h_x^{-1} = \text{Pr}_x \circ \chi_{\mathcal{H}} \upharpoonright \text{Im}(h_x), \\ \text{Im}(h_x) = \{\theta(\iota_x(f_x)) \upharpoonright \mathcal{H} \mid f_x \in \mathcal{L}(\mathfrak{E}_x)\}. \end{cases}$$

PROOF. $\forall y \in X$ we have

$$\text{Pr}_y \circ \theta(\iota_x(f_x)) = \underset{y}{\text{Pr}}(\iota_x(f_x)) \circ \underset{y}{\text{Pr}} = \begin{cases} \mathbf{0}_y, x \neq y \\ f_x \circ \text{Pr}_x, x = y. \end{cases}$$

Moreover

$$\text{Pr}_y \circ g_x(f_x) = \underset{y}{\text{Pr}} \circ \iota_x \circ f_x \circ \underset{x}{\text{Pr}} = \begin{cases} \mathbf{0}_y, x \neq y \\ f_x \circ \text{Pr}_x, x = y. \end{cases}$$

So the first sentence of statement (1) follows. Thus $h_x(\mathcal{L}(\mathfrak{E}_x)) = g_x(\mathcal{L}(\mathfrak{E}_x)) \upharpoonright \mathcal{H} = \theta(\iota_x(\mathcal{L}(\mathfrak{E}_x))) \upharpoonright \mathcal{H}$ so by (6b) of Notations 3.3.3 the second sentence of statement (1) follows. Statement (2) follows by the trivial linearity of g_x and by the second sentence of statement (1).

Let $f_x \in \mathcal{L}(\mathfrak{E}_x)$ and $w = \iota_x \circ f_x \circ \text{Pr}_x \upharpoonright \mathcal{H}$. Then by the assumption (6) we have that $w \in \text{End}(\mathcal{H})$, and $\chi_{\mathcal{H}}(w) = \iota_x(f_x)$, indeed $\text{Pr}_x(\chi_{\mathcal{H}}(w)) = \text{Pr}_x \circ \iota_x \circ f_x \circ \text{Pr}_x \circ \iota_x = f_x = \text{Pr}_x(\iota_x(f_x))$. Thus $\iota_x(f_x) \in \text{Im}(\chi_{\mathcal{H}})$ so by Pr. 3.3.15 $\theta(\iota_x(f_x)) \upharpoonright \mathcal{H} \in \text{Dom}(\chi_{\mathcal{H}})$ and h_x^{-1} is well set. Moreover

$$\begin{aligned} (\underset{x}{\text{Pr}} \circ \chi_{\mathcal{H}}) \circ h_x(f_x) &= \underset{x}{\text{Pr}} \circ \chi_{\mathcal{H}} \circ \theta_{\mathcal{H}}(\iota_x(f_x)) \\ &= \underset{x}{\text{Pr}}(\iota_x(f_x)) = f_x, \end{aligned}$$

where the first equality comes by stat. (1) and by $\iota_x(f_x) \in \text{Im}(\chi_{\mathcal{H}})$, while the second by Prop. 3.3.15. Finally

$$\begin{aligned} g_x \circ \text{Pr}_x \circ \chi_{\mathcal{H}}(\theta(\iota_x(f_x))) &= g_x \circ \text{Pr}_x(\iota_x(f_x)) \\ &= g_x(f_x) = \theta(\iota_x(f_x)). \end{aligned}$$

Thus stat. (3) follows. \square

LEMMA 3.3.17. *If $\langle \mathcal{L}(\mathfrak{E}_x), \tau_x \rangle$ is a topological algebra for all $x \in X$ and \mathcal{A} is an algebra then \mathfrak{G} is a topological algebra.*

PROOF. Let us set for all $F \in \mathfrak{G}$ $L_F : \mathfrak{G} \ni H \mapsto F \circ H \in \mathfrak{G}$, well set \mathfrak{G} being an algebra by Lemma 3.3.14. Thus for all $x \in X$, $f \in \mathcal{A}$ and $l_x \in \mathcal{L}(\mathfrak{E}_x)$

$$\begin{aligned} (L_{\theta(f)} \circ h_x)l_x &= L_{\theta(f)}(\theta(\iota_x(l_x)) \upharpoonright \mathcal{H}) = \theta(f \circ \iota_x(l_x)) \upharpoonright \mathcal{H} \\ &= [\theta \circ \iota_x(f(x) \circ l_x)] \upharpoonright \mathcal{H} = [g_x(f(x) \circ l_x)] \upharpoonright \mathcal{H} \\ &= h_x(f(x) \circ l_x) = (h_x \circ L_{f(x)})l_x, \end{aligned}$$

where $L_{f_x} : \mathcal{L}(\mathfrak{E}_x) \ni s_x \mapsto f_x \circ s_x \in \mathcal{L}(\mathfrak{E}_x)$ for all $f_x \in \mathcal{L}(\mathfrak{E}_x)$. Here the first and fourth equality follow by Prop. 3.3.16, the second one by Lemma 3.3.14. Moreover by hypothesis $L_{f(x)}$ is continuous, while h_x is continuous by [TVS, Prop.5, No4, §4 Ch 2], so $L_{\theta(f)} \circ h_x$ is linear and continuous. Therefore $L_{\theta(f)}$ is linear and continuous by [TVS, Prop. 5, No4, §4 Ch 2]. Similarly R_F is linear and continuous, where $R_F : \mathfrak{G} \ni H \mapsto H \circ F \in \mathfrak{G}$, thus the statement. \square

DEFINITION 3.3.18. *Set*

$$\begin{cases} \Psi_Y^{\mathcal{H}} : \text{End}(\mathcal{H})^Y \rightarrow \prod_{x \in X} \text{End}(\mathfrak{E}_x)^Y, \\ (\text{Pr}_x \circ \Psi_Y^{\mathcal{H}})(\overline{F})(s) = (\text{Pr}_x \circ \chi_{\mathcal{H}})(\overline{F}(s)). \end{cases}$$

Moreover set

$$\begin{cases} \Lambda : \prod_{x \in X} \text{End}(\mathfrak{E}_x)^Y \rightarrow (\text{End}(\prod_{x \in X} \mathfrak{E}_x))^Y, \\ \Lambda(F)(s) = \theta(F(\cdot)(s)). \end{cases}$$

$\forall \overline{F} \in \text{End}(\mathcal{H})^Y$, $\forall F \in \prod_{x \in X} \text{End}(\mathfrak{E}_x)^Y$, $\forall x \in X$ and $\forall s \in Y$, where $F(\cdot)(s) \in \prod_{y \in X} \text{End}(\mathfrak{E}_x)$ such that $\text{Pr}_x(F(\cdot)(s)) = F(x)(s)$.

Finally set

$$\Lambda_{\mathcal{A}}^Y \doteq \Lambda \upharpoonright \left\{ F \in \prod_{x \in X} \mathcal{L}(\mathfrak{E}_x)^Y \mid (\forall s \in Y)(F(\cdot)(s) \in \mathcal{A}) \right\}.$$

PROPOSITION 3.3.19. *Let $x \in X$ and $s \in Y$, then for all $\overline{F} \in \text{End}(\mathcal{H})^Y$*

- (1) $(\text{Pr}_x \circ \Psi_Y^{\mathcal{H}})(\overline{F})(s) = \text{Pr}_x \circ \overline{F}(s) \circ \iota_x$;
- (2) $\Psi_Y^{\mathcal{H}} \circ \Lambda_{\mathcal{A}}^Y = \text{Id}$;
- (3) $\text{Im}(\Lambda_{\mathcal{A}}^Y) \subset G^Y$.

PROOF. Stats. (1) and (3) are trivial. Let $F \in \text{Dom}(\Lambda_{\mathcal{A}}^Y)$ so

$$\begin{aligned} (\text{Pr}_x \circ \Psi_Y^{\mathcal{H}} \circ \Lambda_{\mathcal{A}}^Y)(F)(s) &= (\text{Pr}_x \circ \chi_{\mathcal{H}})(\Lambda_{\mathcal{A}}^Y(F)(s)) = \text{Pr}_x \circ \Lambda_{\mathcal{A}}^Y(F)(s) \circ \iota_x \\ &= \text{Pr}_x \circ \theta(F(\cdot)(s)) \circ \iota_x = \text{Pr}_x(F(\cdot)(s)) \circ \text{Pr}_x \circ \iota_x \\ &= F(x)(s) = \text{Pr}_x(F)(s), \end{aligned}$$

and stat. (2) follows. □

PROPOSITION 3.3.20. $(\forall x \in X)(\forall s \in Y)(\forall \overline{F} \in G^Y)$ we have

$$(\text{Pr}_x \circ \Psi_Y^{\mathcal{H}})(\overline{F})(s) \circ \text{Pr}_x = \text{Pr}_x \circ (\overline{F}(s))$$

PROOF. Let $\overline{F} \in G^Y$ thus $\exists U \in \mathcal{A}^Y$ such that $\overline{F}(s) = \theta(U(s)) \upharpoonright \mathcal{H}$, hence for all $x \in X, s \in Y$

$$\begin{aligned} (\text{Pr}_x \circ \Psi_Y^{\mathcal{H}})(\overline{F})(s) \circ \text{Pr}_x &= \text{Pr}_x(\Psi_Y^{\mathcal{H}}(\overline{F}))(s) \circ \text{Pr}_x \\ &= \text{Pr}_x \circ \overline{F}(s) \circ \iota_x \circ \text{Pr}_x, && \text{by Prop. 3.3.19} \\ &= (\text{Pr}_x \circ \theta(U(s))) \upharpoonright \mathcal{H} \circ \iota_x \circ \text{Pr}_x \\ &\doteq (\text{Pr}_x(U(s)) \circ \text{Pr}_x) \upharpoonright \mathcal{H} \circ \iota_x \circ \text{Pr}_x \\ &= \text{Pr}_x(U(s)) \circ \text{Pr}_x \upharpoonright \mathcal{H} \\ &\doteq \text{Pr}_x \circ \theta(U(s)) \upharpoonright \mathcal{H} \\ &= \text{Pr}_x \circ (\overline{F}(s)). \end{aligned}$$

□

DEFINITION 3.3.21. *Let $x \in X$*

$$\begin{cases} I_x : Hom(\mathcal{L}(\mathfrak{E}_x), \mathbb{K}) \rightarrow Hom\left(\prod_{y \in X} \mathcal{L}(\mathfrak{E}_y), \mathbb{K}\right), \\ I_x(t_x) \doteq t_x \circ \text{Pr}_x. \end{cases}$$

LEMMA 3.3.22. *Let $x \in X$ thus*

(1) $(\forall t_x \in Hom(\mathcal{L}(\mathfrak{E}_x), \mathbb{K}))(\forall y \in X)$ *we have*

$$\begin{cases} I_x(t_x) \circ \chi_{\mathcal{H}} \circ h_y = t_x, x = y \\ I_x(t_x) \circ \chi_{\mathcal{H}} \circ h_y = \mathbf{0}, x \neq y; \end{cases}$$

(2) $(\forall t_x \in \langle \mathcal{L}(\mathfrak{E}_x), \tau_x \rangle^*)(I_x(t_x) \circ \chi_{\mathcal{H}} \upharpoonright \mathfrak{G} \in \mathfrak{G}^*)$

PROOF. Let $x \in X$ and $t_x \in Hom(\mathcal{L}(\mathfrak{E}_x), \mathbb{K})$ thus for all $y \in X$ and $f_y \in \mathcal{L}(\mathfrak{E}_y)$ we have

$$\begin{aligned} I_x(t_x) \circ \chi_{\mathcal{H}} \circ h_y(f_y) &= t_x \circ \text{Pr}_x \circ \chi_{\mathcal{H}}(\iota_y \circ f_y \circ \text{Pr}_y \upharpoonright \mathcal{H}) \\ &= t_x \circ (\text{Pr}_x \circ \iota_y \circ f_y \circ \text{Pr}_y \circ \iota_x), \end{aligned}$$

and stat. (1) follows. Stat. (2) follows by stat. (1) and [TVS, Prop. 5, No4, §4 Ch 2]. \square

The following is the first main result of this section.

THEOREM 3.3.23. *We have*

(1) $\Psi_Y^{\mathcal{H}} \in Hom(\mathfrak{L}_1(Y, \mathfrak{G}, \mu), \prod_{x \in X} \mathfrak{L}_1(Y, \langle \mathcal{L}(\mathfrak{E}_x), \tau_x \rangle, \mu));$

(2) $(\forall x \in X)(\forall s \in Y)(\forall \overline{F} \in \mathfrak{L}_1(Y, \mathfrak{G}, \mu))$

$$\int \text{Pr}_x(\Psi_Y^{\mathcal{H}}(\overline{F}))(s) d\mu(s) = \text{Pr}_x \circ \left[\int \overline{F}(s) d\mu(s) \right] \circ \iota_x.$$

PROOF. Let $x \in X$, set

$$\Delta_x : G \ni f \mapsto \text{Pr}_x \circ f \circ \iota_x \in \mathcal{L}(\mathfrak{E}_x).$$

Δ_x is well-defined by Lemma 3.3.14. By applying [TVS, Prop. 5, No3, §4 Ch 2] $\Delta_x \in \mathcal{L}(\mathfrak{G}, \langle \mathcal{L}(\mathfrak{E}_x), \tau_x \rangle)$ if and only if $(\forall y \in X)(\Delta_x \circ h_y \in \mathcal{L}(\langle \mathcal{L}(\mathfrak{E}_y), \tau_y \rangle, \langle \mathcal{L}(\mathfrak{E}_x), \tau_x \rangle))$. Moreover $\forall y \in X$ and $\forall f_y \in \mathcal{L}(\mathfrak{E}_y)$ we have

$$(\Delta_x \circ h_y)(f_y) = \text{Pr}_x \circ \iota_y \circ f_y \circ \text{Pr}_y \circ \iota_x,$$

so

$$\begin{cases} \Delta_x \circ h_y = Id, x = y \\ \Delta_x \circ h_y = \mathbf{0}, x \neq y. \end{cases}$$

In any case $\Delta_x \circ h_y \in \mathcal{L}(\langle \mathcal{L}(\mathfrak{E}_y), \tau_y \rangle, \langle \mathcal{L}(\mathfrak{E}_x), \tau_x \rangle)$, thus

$$\Delta_x \in \mathcal{L}(\mathfrak{G}, \langle \mathcal{L}(\mathfrak{E}_x), \tau_x \rangle)$$

hence

$$(3.3.10) \quad (\forall t_x \in \langle \mathcal{L}(\mathfrak{E}_x), \tau_x \rangle^*)(t_x \circ \Delta_x \in \mathfrak{G}^*).$$

Therefore

$$\begin{aligned} t_x \left(\Pr_x \circ \left[\int \overline{F}(s) d\mu(s) \right] \circ \iota_x \right) &= (t_x \circ \Delta) \left(\int \overline{F}(s) d\mu(s) \right) \\ &= \int (t_x \circ \Delta)(\overline{F}(s)) d\mu(s) \\ &= \int t_x \left(\Pr_x \circ \overline{F}(s) \circ \iota_x \right) d\mu(s) \\ &= \int t_x \left((\Pr_x \circ \Psi_Y^{\mathcal{H}})(\overline{F})(s) \right) d\mu(s), \end{aligned}$$

where the second equality comes by (3.3.10) and [INT, Prop.1, No1, §1, Ch. 6], while the last one comes by Proposition 3.3.19. \square

DEFINITION 3.3.24. *Let Z be a topological vector space set*

$$\begin{cases} \mathbf{ev}_Z \in \text{Hom}(Z, \text{Hom}(\mathcal{L}(Z), Z)), \\ (\forall v \in Z)(\forall f \in \mathcal{L}(Z))(\mathbf{ev}_Z(v)(f)) \doteq f(v). \end{cases}$$

Moreover set $\eta \doteq \mathbf{ev}_{\mathcal{H}}$ and $\forall x \in X$ set $\varepsilon_x \doteq \mathbf{ev}_{\mathfrak{E}_x}$.

LEMMA 3.3.25. *Let $D \subseteq \mathcal{H}$ thus $(A) \Rightarrow (B)$, where*

- (A): $(\forall x \in X)(\forall v_x \in \Pr_x(D))(\varepsilon_x(v_x) \in \mathcal{L}(\langle \mathcal{L}(\mathfrak{E}_x), \tau_x \rangle, \mathfrak{E}_x));$
- (B): $(\forall v \in D)(\eta(v) \in \mathcal{L}(\mathfrak{G}, \langle \mathcal{H}, \mathfrak{T} \rangle)).$

PROOF. Let $y \in X$ thus for all $v \in \mathcal{H}$

$$\eta(v) \circ h_y = \iota_y \circ \varepsilon_y(\Pr_y(v)).$$

Hence by (A) and the fact that by construction ι_y is continuous with respect to the topology \mathfrak{T} we have for all $v \in D$

$$\eta(v) \circ g_y \in \mathcal{L}(\langle \mathcal{L}(\mathfrak{E}_y), \tau_y \rangle, \langle \mathcal{H}, \mathfrak{T} \rangle).$$

Thus (B) follows by the universal property of any locally final topology, cf. [TVS, (ii) of Prop. 5, N 4, §4 Ch 2]. \square

The following is the second main result of the section

THEOREM 3.3.26. *Let $D \subseteq \mathcal{H}$ and assume (A) of Lemma 3.3.25. Then $(\forall \overline{F} \in \mathfrak{L}_1(Y, \mathfrak{G}, \mu))(\forall x \in X)(\forall v \in D)$*

$$(3.3.11) \quad \int \left\langle \text{Pr}_x(\Psi_Y^{\mathcal{H}}(\overline{F}))(s), v(x) \right\rangle_x d\mu(s) = \left\langle \int \overline{F}(s) d\mu(s), v \right\rangle(x).$$

Here the integral in the left-side is with respect to the μ and the lct on \mathfrak{E}_x , while the integral in the right-side is with respect to the μ and the lct on \mathfrak{G} .

PROOF. $(\forall \overline{F} \in \mathfrak{L}_1(Y, \mathfrak{G}, \mu))(\forall x \in X)(\forall v \in D)$ we have

$$\begin{aligned} \text{Pr}_x \circ \left[\int \overline{F}(s) d\mu(s) \right] (v) &= (\text{Pr}_x \circ \eta(v)) \left(\int \overline{F}(s) d\mu(s) \right) \\ &= \int (\text{Pr}_x \circ \eta(v))(\overline{F}(s)) d\mu(s) \\ &= \int (\text{Pr}_x \circ \overline{F}(s))(v) d\mu(s) \\ &= \int \text{Pr}_x(\Psi_Y^{\mathcal{H}}(\overline{F}))(s)(v(x)) d\mu(s). \end{aligned}$$

Here in the second equality we applied [INT, Prop.1, No1, §1, Ch. 6] and the fact that $\text{Pr}_x \circ \eta(v) \in \mathcal{L}(\mathfrak{G}, \mathfrak{E}_x)$ because of Lemma 3.3.25 and the linearity and continuity of Pr_x with respect to the topology \mathfrak{T} . Finally in the last equality we used Prop. 3.3.20. \square

The following is the main result of this section

THEOREM 3.3.27. *Let $D \subseteq \mathcal{H}$ and assume (A) of Lemma 3.3.25. Then $(\forall \overline{F} \in \mathfrak{L}_1(Y, \mathfrak{G}, \mu))(\forall x \in X)(\forall v \in D)$*

$$(3.3.12) \quad \boxed{\left\langle \int \text{Pr}_x(\Psi_Y^{\mathcal{H}}(\overline{F}))(s) d\mu(s), v(x) \right\rangle_x = \left\langle \int \overline{F}(s) d\mu(s), v \right\rangle(x).}$$

Equivalently \mathfrak{G} is a \mathbf{U} -space with respect to $\{\langle \mathcal{L}(\mathfrak{E}_x), \tau_x \rangle\}_{x \in X}$, \mathfrak{T} and D . Here the integral in the left-side is with respect to the μ and the lct on $\langle \mathcal{L}(\mathfrak{E}_x), \tau_x \rangle$.

PROOF. By (A) of Lemma 3.3.25, stat.(1) of Th. 3.3.23 and [INT, Prop.1, No1, §1, Ch. 6] we have $(\forall \bar{F} \in \mathfrak{L}_1(Y, \mathfrak{G}, \mu))(\forall x \in X)(\forall v \in D)$

$$\int \left\langle \text{Pr}_x(\Psi_Y^{\mathcal{H}}(\bar{F}))(s), v(x) \right\rangle_x d\mu(s) = \left\langle \int_x \text{Pr}_x(\Psi_Y^{\mathcal{H}}(\bar{F}))(s) d\mu(s), v(x) \right\rangle_x,$$

hence the statement follows by Theorem 3.3.26. \square

REMARK 3.3.28. By (3.3.12) and stat.(2) of Th. 3.3.23 $(\forall \bar{F} \in \mathfrak{L}_1(Y, \mathfrak{G}, \mu))(\forall x \in X)(\forall v \in D)$

$$\left\langle \int \bar{F}(s) d\mu(s), v \right\rangle(x) = \left\langle \int \bar{F}(s) d\mu(s), \iota_x(v(x)) \right\rangle(x).$$

Thus for all $v, w \in D$ and $x \in X$

$$v(x) = w(x) \Rightarrow \left\langle \int \bar{F}(s) d\mu(s), v \right\rangle(x) = \left\langle \int \bar{F}(s) d\mu(s), w \right\rangle(x).$$

COROLLARY 3.3.29. Let $\mathcal{S} \in \prod_{x \in X} 2^{\text{Bounded}(\mathfrak{E}_x)}$ and \mathcal{D} such that

$$(3.3.13) \quad \begin{cases} N(x) \doteq \bigcup_{l_x \in L_x} B_{l_x}^x \text{ is total in } \mathfrak{E}_x, \forall x \in X, \\ \mathcal{D} \subseteq \mathcal{H} \cap \prod_{x \in X} N(x), \end{cases}$$

where $\mathcal{S}(x) = \{B_{l_x}^x \mid l_x \in L_x\}$. Assume that for all $x \in X$ the topology τ_x is generated by the set of seminorms $\{p_{(l_x, j_x)}^x \mid (l_x, j_x) \in L_x \times J_x\}$, where ⁴

$$(3.3.14) \quad p_{(l_x, j_x)}^x : \mathcal{L}(\mathfrak{E}_x) \ni f_x \mapsto \sup_{w \in B_{l_x}^x} \nu_{j_x}^x(f_x w) \in \mathbb{R}^+.$$

Then

- (1) (A) of Lemma 3.3.25 for $D = \mathcal{D}$;
- (2) (3.3.11) holds and \mathfrak{G} is a \mathbf{U} -space with respect to $\{\mathcal{L}_{\mathcal{S}(x)}(\mathfrak{E}_x)\}_{x \in X}$, \mathfrak{T} and \mathcal{D} .

PROOF. By request (3.3.13) we have that the lcs $\langle \mathcal{L}(\mathfrak{E}_x), \tau_x \rangle$ is Hausdorff so the position is well-set. By construction $(\forall x \in X)(\forall v_x \in D(x))(\exists \bar{l}_x \in L_x)(v_x \in B_{\bar{l}_x}^x)$, so $(\forall f_x \in \mathcal{L}(\mathfrak{E}_x))(\forall j_x \in J_x)$

$$\begin{aligned} \nu_{j_x}^x(\varepsilon_x(v_x)f_x) &= \nu_{j_x}^x(f_x(v_x)) \\ &\leq p_{(\bar{l}_x, j_x)}^x(f_x), \end{aligned}$$

hence statement (1) by [TVS, Prop. 5, No4, §1 Ch 2]. Statement (2) follows by statement (1), Theorem 3.3.26 and Theorem 3.3.27 respectively. \square

⁴In others words $\langle \mathcal{L}(\mathfrak{E}_x), \tau_x \rangle = \mathcal{L}_{\mathcal{S}_x}(\mathfrak{E}_x)$, see Notations 1.1 and Def. 2.1.2.

COROLLARY 3.3.30 ($\mathbf{LD}(\mathcal{O}, \mathcal{D})$). Let $\langle \mathfrak{V}, \mathfrak{W}, X, \mathbb{R}^+ \rangle$ be a (Θ, \mathcal{E}) -structure satisfying (3.1.14) and $\Gamma(\pi) \cap \mathcal{H} \cap \prod_{x \in X} \mathcal{B}_B^x \neq \emptyset$. Set

$$(3.3.15) \quad \begin{cases} \mathcal{O} \subseteq \Gamma(\rho) \\ \mathcal{D} \subseteq \Gamma(\pi) \cap \mathcal{H} \cap \prod_{x \in X} \mathcal{B}_B^x \end{cases}$$

If $\exists \mathcal{F} \subset \bigcap_{\lambda > 0} \mathfrak{L}_1(\mathbb{R}^+, \mathfrak{G}, \mu_\lambda)$ such that $\Psi_{\mathbb{R}^+}^{\mathcal{H}}(\mathcal{F}) = \mathcal{O}$ then (3.3.9) holds.

In particular if $\exists \mathcal{F} \subset \bigcap_{\lambda > 0} \mathfrak{L}_1(\mathbb{R}^+, \mathfrak{G}, \mu_\lambda)$ such that $\Psi_{\mathbb{R}^+}^{\mathcal{H}}(\mathcal{F}) = \mathcal{O}$ then

$$\langle \mathfrak{G}, \mathcal{D} \rangle \subseteq \Gamma(\pi) \Rightarrow \mathbf{LD}(\mathcal{O}, \mathcal{D}).$$

Here \mathcal{B}_B^x , for all $x \in X$, is defined in (2.2.1).

PROOF. By statement (2) of Cor. 3.3.29, Pr. 3.3.7 and Rm. 3.3.8. \square

REMARK 3.3.31. Note that if $\mathcal{E} \subset \Theta$, as for example for the positions taken in Rm. 3.1.12, we have $\mathcal{E} \subset \prod_{x \in X} \mathcal{B}_B^x$. Hence if $\mathcal{E} \subseteq \mathcal{H}$ we have $\mathcal{E} \subseteq \Gamma(\pi) \cap \mathcal{H} \cap \prod_{x \in X} \mathcal{B}_B^x$.

By the previous remark, Cor. 3.3.30 and Thm. 3.1.16 we can state

COROLLARY 3.3.32. Let us assume the hypotheses of Thm. 3.1.16 made exception for the (i) replaced by the following one: $\mathcal{E} \subseteq \mathcal{H}$ and $\exists \mathcal{F} \subset \bigcap_{\lambda > 0} \mathfrak{L}_1(\mathbb{R}^+, \mathfrak{G}, \mu_\lambda)$ such that $\Psi_{\mathbb{R}^+}^{\mathcal{H}}(\mathcal{F}) = \Gamma(\rho)$ and

$$\langle \mathfrak{G}, \mathcal{E} \rangle \subseteq \Gamma(\pi).$$

Then all the statements of Thm. 3.1.16 hold

3.3.2. Uniform Convergence over $\mathcal{K} \in \text{Compact}(\langle \mathcal{H}, \mathfrak{T} \rangle)$. In this subsection we assume given the following data

- (1) a \mathfrak{V} Banach bundle, a (Θ, \mathcal{E}) -structure $\langle \mathfrak{V}, \mathfrak{M}, X, Y \rangle$ where Θ is defined in (3.1.16), where we denote $\mathfrak{W} \doteq \langle \langle \mathfrak{M}, \gamma \rangle, \rho, X, \mathfrak{R} \rangle$ and $\mathfrak{V} \doteq \langle \langle \mathfrak{E}, \tau \rangle, \pi, X, \{\|\cdot\|\} \rangle$;
- (2) a Banach space $\langle \mathcal{H}, \|\cdot\|_{\mathcal{H}} \rangle$ such that $\langle \mathcal{H}, \mathfrak{T} \rangle$ satisfies (6) of Notations 3.3.3, where \mathfrak{T} is the topology induced by the norm $\|\cdot\|_{\mathcal{H}}$ and τ_x is such that $\langle \mathcal{L}(\mathfrak{E}_x), \tau_x \rangle = \mathcal{L}_{S_x}(\mathfrak{E}_x)$ for every $x \in X$;
- (3) \mathcal{A} as in (6) of Notations 3.3.3;
- (4) \mathfrak{G} , $\Psi_Y^{\mathcal{H}}$ and $\Lambda_{\mathcal{A}}^Y$ as defined in Def. 3.3.12 and Def. 3.3.18 respectively.

The proof of the following Lemma is an adaptation to the present framework of the proof of [Ch, Prop. 5.13].

LEMMA 3.3.33. Let $\mathcal{U} \in \prod_{x \in X} \mathfrak{M}_x$ and $x_\infty \in X$ moreover assume that

- (1) $\mathcal{E} \subseteq \mathcal{H} \subseteq \prod_{x \in X}^b \mathfrak{E}_x$ such that $(\exists a > 0)(\forall f \in \mathcal{H})(\|f\|_{\sup} \leq a\|f\|_{\mathcal{H}})$, where $\|f\|_{\sup} \doteq \sup_{x \in X} \|f(x)\|_x$;
- (2) $\exists F \in \Gamma(\rho)$ such that $F(x_\infty) = \mathcal{U}(x_\infty)$ and $\{F(\cdot)(s) \mid s \in Y\} \subseteq \mathcal{A}$
- (3) $\{\mathcal{U}(\cdot)(s) \mid s \in Y\} \subseteq \mathcal{A}$;
- (4) $\{\overline{F}(s) \mid s \in Y\}$ and $\{\overline{\mathcal{U}}(s) \mid s \in Y\}$ are equicontinuous as subsets of $\mathcal{L}(\langle \mathcal{H}, \|\cdot\|_{\mathcal{H}} \rangle)$, where $\overline{\mathcal{U}} \doteq \Lambda_{\mathcal{A}}^Y(\mathcal{U})$. and $\overline{F} \doteq \Lambda_{\mathcal{A}}^Y(F)$.

Then (A) \Leftrightarrow (B) where

(A): $\mathcal{U} \in \Gamma^{x_\infty}(\rho)$;

(B): For all $\mathcal{K} \in \text{Compact}(\mathcal{H})$ such that $\mathcal{K} \subseteq \mathcal{E}$ and for all $K \in \text{Compact}(Y)$

$$\lim_{z \rightarrow x_\infty} \sup_{s \in K} \sup_{v \in \mathcal{K}} \|\mathcal{U}(z)(s)v(z) - F(z)(s)v(z)\| = 0.$$

PROOF. We shall prove only (A) \Rightarrow (B), indeed the other implication follows by (3) \Rightarrow (4) of Lemma 2.2.5. So assume (A) to be true. In this proof let us set $B(\mathcal{H}) \doteq \mathcal{L}(\langle \mathcal{H}, \|\cdot\|_{\mathcal{H}} \rangle)$, moreover $\Psi \doteq \Psi_Y^{\mathcal{H}}$ and $\Lambda \doteq \Lambda_{\mathcal{A}}^Y$, moreover set $\overline{F} \doteq \Lambda_{\mathcal{A}}^Y(F)$ for every $F \in \Gamma(\rho)$; thus by stat. (2) of Pr. 3.3.19 $\Psi(\overline{F}) = F$ and $\Psi(\overline{\mathcal{U}}) = \mathcal{U}$. Hence by Pr. 3.3.20 for all $v \in \mathcal{E}$ $F \in \Gamma(\rho)$, $z \in X$ and $s \in Y$

$$(3.3.16) \quad \mathcal{U}(z)(s)v(z) = (\overline{\mathcal{U}}v)(z), \quad F(z)(s)v(z) = (\overline{F}v)(z).$$

By (A) and implication (4) \Rightarrow (3) of Lemma 2.2.5 we have for all $K \in \text{Compact}(Y)$ and $v \in \mathcal{E}$

$$(3.3.17) \quad \lim_{z \rightarrow x_\infty} \sup_{s \in K} \|\mathcal{U}(z)(s)v(z) - F(z)(s)v(z)\| = 0.$$

Fix $\mathcal{K} \in \text{Compact}(\mathcal{H})$ such that $\mathcal{K} \subseteq \mathcal{E}$, $f \in \mathcal{K}$ and $\varepsilon > 0$, thus by (3.3.17) and (3.3.16) there exists U neighbourhood of x_∞ such that

$$(3.3.18) \quad \sup_{s \in K} \sup_{z \in U} \|[(\overline{\mathcal{U}}(s) - \overline{F}(s))f](z)\| \leq \varepsilon/2.$$

Define

$$\begin{cases} M \doteq \max\{\sup_{s \in Y} \|\overline{F}(s)\|_{B(\mathcal{H})}, \sup_{s \in Y} \|\overline{\mathcal{U}}(s)\|_{B(\mathcal{H})}\} \\ \eta \doteq \varepsilon/4aM \\ \mathfrak{U}(f) \doteq \{g \in \mathcal{K} \mid \|f - g\|_{\mathcal{H}} < \eta\}. \end{cases}$$

Thus for all $g \in \mathfrak{U}(f)$

$$\begin{aligned}
& \sup_{z \in U} \sup_{s \in K} \|\mathcal{U}(z)(s)g(z) - F(z)(s)g(z)\| = \\
& \sup_{s \in K} \sup_{z \in U} \|[(\overline{\mathcal{U}}(s) - \overline{F}(s))g](z)\| \leq \\
& \sup_{s \in K} \sup_{z \in U} \|[(\overline{\mathcal{U}}(s) - \overline{F}(s))f](z)\| + \sup_{s \in K} \sup_{z \in U} \|\overline{\mathcal{U}}(s)(g - f)(z)\| + \sup_{s \in K} \sup_{z \in U} \|F(s)(g - f)(z)\| \leq \\
& \varepsilon/2 + a \sup_{s \in K} \|\overline{\mathcal{U}}(s)(g - f)\|_{\mathcal{H}} + a \sup_{s \in K} \|F(s)(g - f)\|_{\mathcal{H}} \leq \\
& \varepsilon/2 + 2aM \|g - f\|_{\mathcal{H}} < \varepsilon.
\end{aligned}$$

Therefore (B) follows by considering that $\{\mathfrak{U}(f) \mid f \in \mathcal{K}\}$ is an open cover of the compact \mathcal{K} . Indeed let for example $\{\mathfrak{U}(f_i) \mid i = 1, \dots, n\}$ a finite subcover of \mathcal{K} thus by setting $W \doteq \bigcap_{i=1}^n U_n$ with obvious meaning of U_i , we have

$$\sup_{z \in W} \sup_{s \in K} \sup_{g \in \mathcal{K}} \|\mathcal{U}(z)(s)g(z) - F(z)(s)g(z)\| < \varepsilon.$$

□

REMARK 3.3.34. We can set $\mathcal{H} = \prod_{x \in X}^b \mathfrak{E}_x$ with the usual norm $\|\cdot\|_{\text{sup}}$.

THEOREM 3.3.35 (\mathcal{K} –Uniform Convergence). *Let $\mathfrak{V} \doteq \langle \langle \mathfrak{E}, \tau \rangle, \pi, X, \|\cdot\| \rangle$ be a Banach bundle. Let $x_\infty \in X$ and $\mathcal{U}_0 \in \prod_{x \in X_0} \mathcal{C}(\mathbb{R}^+, B_s(\mathfrak{E}_x))$ be such that $\mathcal{U}_0(x)$ is a (C_0) –semigroup of contractions (respectively of isometries) on \mathfrak{E}_x for all $x \in X_0$. Assume that*

- (1) $D(T_{x_\infty})$ is dense in \mathfrak{E}_{x_∞} ;
- (2) \mathfrak{V} and \mathfrak{W} satisfy (3.1.18);
- (3) $\exists \lambda_0 > 0$ (respectively $\exists \lambda_0 > 0, \lambda_1 < 0$) such that the range $\mathcal{R}(\lambda_0 - T_{x_\infty})$ is dense in \mathfrak{E}_{x_∞} , (respectively the ranges $\mathcal{R}(\lambda_0 - T_{x_\infty})$ and $\mathcal{R}(\lambda_1 - T_{x_\infty})$ are dense in \mathfrak{E}_{x_∞});
- (4) $\mathbf{U}_{\|\cdot\|_{B(\mathfrak{E}_z)}}(\mathcal{L}_{S_z}(\mathfrak{E}_z)) \subseteq \mathfrak{M}_z$ (respectively $\mathbf{U}_{is}(\mathcal{L}_{S_z}(\mathfrak{E}_z)) \subseteq \mathfrak{M}_z$) for all $z \in X$;
- (5) $\mathcal{E} \subseteq \mathcal{H} \subseteq \prod_{x \in X}^b \mathfrak{E}_x$
- (6) X is metrizable;
- (7) $\exists \mathcal{F} \subset \bigcap_{\lambda > 0} \mathfrak{L}_1(\mathbb{R}^+, \mathfrak{G}, \mu_\lambda)$ such that $\Psi_{\mathbb{R}^+}^{\mathcal{H}}(\mathcal{F}) = \Gamma(\rho)$;
- (8) $(\exists F \in \Gamma(\rho))(F(x_\infty) = \mathcal{U}(x_\infty))$ such that
 - (a) $\langle \int \overline{F}(s) d\mu_\lambda(s), \mathcal{E} \rangle \subseteq \Gamma(\pi)$, for all $\lambda > 0$;

- (b) $(\forall v \in \mathcal{E})(\exists \phi \in \Phi)$ s.t. $\phi_1(x_\infty) = v(x_\infty)$ and $(\forall \{z_n\}_{n \in \mathbb{N}} \subset X \mid \lim_{n \in \mathbb{N}} z_n = x_\infty)$ we have that $\{\mathcal{U}(z_n)(\cdot)\phi_1(z_n) - F(z_n)(\cdot)v(z_n)\}_{n \in \mathbb{N}}$ is a bounded equicontinuous sequence.

Then

$$(3.3.19) \quad \mathcal{U} \in \Gamma^{x_\infty}(\rho).$$

Furthermore if

- (1) $(\exists a > 0)(\forall f \in \mathcal{H})(\|f\|_{\sup} \leq a\|f\|_{\mathcal{H}})$,
- (2) $\{F(\cdot)(s) \mid s \in \mathbb{R}^+\} \subseteq \mathcal{A}$ and $\{\mathcal{U}(\cdot)(s) \mid s \in \mathbb{R}^+\} \subseteq \mathcal{A}$;
- (3) $\{\overline{F}(s) \mid s \in \mathbb{R}^+\}$ and $\{\overline{\mathcal{U}}(s) \mid s \in \mathbb{R}^+\}$ are equicontinuous as subsets of $\mathcal{L}(\langle \mathcal{H}, \|\cdot\|_{\mathcal{H}} \rangle)$.

Then for all $\mathcal{K} \in \text{Compact}(\mathcal{H})$ such that $\mathcal{K} \subseteq \mathcal{E}$ and for all $K \in \text{Compact}(\mathbb{R}^+)$

$$(3.3.20) \quad \lim_{z \rightarrow x_\infty} \sup_{s \in K} \sup_{v \in \mathcal{K}} \|\mathcal{U}(z)(s)v(z) - F(z)(s)v(z)\| = 0.$$

Here $D(T_{x_\infty})$ is defined as in Notations 3.1.1 with \mathcal{T}_0 and Φ given in (3.1.18). While $\mathcal{U} \in \prod_{x \in X} \mathfrak{M}_x$ such that $\mathcal{U} \upharpoonright X_0 \doteq \mathcal{U}_0$ and $\mathcal{U}(x_\infty)$ is the semigroup on \mathfrak{E}_{x_∞} generated by T_{x_∞} operator defined in (3.1.2). Moreover $\|f\|_{\sup} \doteq \sup_{x \in X} \|f(x)\|_x$, while $\overline{\mathcal{U}} \doteq \Lambda_{\mathcal{A}}^Y(\mathcal{U})$ and $\overline{F} \doteq \Lambda_{\mathcal{A}}^Y(F)$.

PROOF. By hyp. (7) and st. (1) of Th. 3.3.23, (3.1.14) follows. Moreover (3.3.15) follows by hyp. (5), and Rm. 3.3.31. Hence by hyps. (7 – 8a), and Cor. 3.3.30 follows the LD($\{F\}, \mathcal{E}$). Then (3.3.19) follows by Th. 3.1.16. (3.3.20) follows by (3.3.19) and Lm. 3.3.33. \square

REMARK 3.3.36. By st.(2) of Pr. 3.3.19 hyp. (7) is equivalent to the following one $\Lambda_{\mathcal{A}}^{\mathbb{R}^+}(\Gamma(\rho)) \subseteq \bigcap_{\lambda > 0} \mathfrak{L}_1(\mathbb{R}^+, \mathfrak{G}, \mu_\lambda)$. In any case the form in hyp. (7) has the advantage to be considered as a tool for constructing $\Gamma(\rho)$. Finally note that

$$\langle \mathfrak{G}, \mathcal{E} \rangle \subseteq \Gamma(\pi) \Rightarrow (8a).$$

3.3.3. $\langle \mathcal{H}, \mathfrak{T} \rangle$ as Direct Integral of a Continuous Field of left-Hilbert and associated left-von Neumann Algebras.

3.3.3.1. Ex1. Assume that $\mathfrak{V} = \langle \langle \mathfrak{E}, \tau \rangle, \pi, X, \mathfrak{N} \rangle$ is a continuous field of left-Hilbert algebras on X . Let \mathcal{H} be the direct integral of \mathfrak{V} with respect to some finite Radon measure on X and $\mathcal{B} \subset \mathcal{H}$ a linear space, set

$$\mathcal{A}(\mathcal{B}) \doteq \{X \ni x \mapsto L_{a(x)} \mid a \in \mathcal{B}\},$$

where $L_{a_x} \in B(\mathfrak{E}_x)$ for any $a_x \in \mathfrak{E}_x$, is the left multiplication on the left-Hilbert algebra \mathfrak{E}_x . Then \mathcal{H} and $\mathcal{A}(\mathcal{B})$ satisfies the requirements in Notations 3.3.3, moreover

$$(3.3.21) \quad \boxed{G(\mathcal{B}) \doteq \theta(\mathcal{A}(\mathcal{B})) \upharpoonright \mathcal{H} = L_{\mathcal{B}},}$$

where $L_a \in B(\mathcal{H})$ for any $a \in \mathcal{H}$, is the left multiplication on the left-Hilbert algebra \mathcal{H} . If every \mathfrak{E}_x is unital then \mathcal{H} is unital, thus $L_{(\cdot)}$ is an injective (isometric) map of \mathcal{H} into $B(\mathcal{H})$. Therefore under this additional requirement we can take the following identification

$$G(\mathcal{B}) \simeq \mathcal{B} \text{ as linear spaces.}$$

Let $\mathbf{H} \doteq \{\mathbf{H}^i \in \prod_{x \in X} \mathfrak{E}_x\}_{i=0}^2$ such that \mathbf{H}_x^0 is a left Hilbert subalgebra of \mathfrak{E}_x , while \mathbf{H}_x^k is a linear subspace of \mathbf{H}_x^0 , for all $k = 1, 2$ and $x \in X$. Set

$$(3.3.22) \quad \begin{cases} \Gamma(\pi, \mathbf{H}) \doteq \{\sigma \in \mathcal{H} \mid (\forall x \in X)(\sigma(x) \in \mathbf{H}_x^0)\} \\ \mathcal{D}_{\mathbf{H}} \doteq \{\sigma \in \mathcal{H} \mid (\forall x \in X)(\sigma(x) \in \mathbf{H}_x^1)\} \\ \mathcal{B}_{\mathbf{H}} \doteq \{\sigma \in \mathcal{H} \mid (\forall x \in X)(\sigma(x) \in \mathbf{H}_x^2)\}. \end{cases}$$

Thus $\Gamma(\pi, \mathbf{H})$ is a left Hilbert subalgebra of \mathcal{H} and $\mathcal{B}_{\mathbf{H}}, \mathcal{D}_{\mathbf{H}}$ are linear subspaces of $\Gamma(\pi, \mathbf{H})$, so

$$(3.3.23) \quad L_{\mathcal{B}_{\mathbf{H}}}(\mathcal{D}_{\mathbf{H}}) \subseteq \Gamma(\pi, \mathbf{H}).$$

By (3.3.23) and (3.3.21) follows that for all $\sigma \in \mathcal{B}_{\mathbf{H}}, \eta \in \mathcal{D}_{\mathbf{H}}$ and $y \in X$

$$(3.3.24) \quad \boxed{\begin{cases} \langle G(\mathcal{B}_{\mathbf{H}}), \mathcal{D}_{\mathbf{H}} \rangle \subseteq \Gamma(\pi, \mathbf{H}), \\ \langle \theta(x \mapsto L_{\sigma(x)}), \eta \rangle(y) = \sigma(y)\eta(y). \end{cases}}$$

3.3.3.2. *Ex2.* Let us consider now the continuous field of left-von Neumann algebras associated to the fixed field of Hilbert algebras, and by abusing of language, let us denote it with the same symbol $\mathfrak{V} = \langle \langle \mathfrak{E}, \tau \rangle, \pi, X, \mathfrak{N} \rangle$, as well as \mathcal{H} will denote the associated direct integral with respect to some finite Radon measure on X . Let Δ_x be the modular operator associated to the Hilbert algebra \mathfrak{E}_x and σ_x the corresponding modular group. Thus we can set

$$\begin{cases} \mathcal{A}_{\Delta} \doteq \{S_t : X \ni x \mapsto \sigma_x(t) \in \text{Aut}(\mathfrak{E}_x) \mid t \in \mathbb{R}\} \\ G_{\Delta} \doteq \theta(\mathcal{A}_{\Delta}) \upharpoonright \mathcal{H} \\ \Sigma_t \doteq \theta(S_t) \upharpoonright \mathcal{H}, t \in \mathbb{R}. \end{cases}$$

Note that for every $t \in \mathbb{R}$, $v \in \mathcal{H}$ and $x \in X$

$$\Sigma_t(v)(x) = \sigma_x(t)(v(x)).$$

Now if we set

$$\Gamma(\pi) \doteq \mathcal{H}$$

for any linear subspace \mathcal{D} of \mathcal{H} we have

$$\boxed{\langle G_\Delta, \mathcal{D} \rangle \subseteq \Gamma(\pi)}.$$

Finally note that to \mathcal{A}_Δ we can associate the following map

$$\overline{\Sigma} : \mathbb{R}^+ \ni t \mapsto \Sigma_t \in G_\Delta,$$

for which we have for all $x \in X$

$$\Psi_{\mathbb{R}}^{\mathcal{H}}(\overline{\Sigma})(x) = \sigma_x.$$

3.3.3.3. *Ex3.* In the previous example we consider the extreme case in which $\Gamma(\pi) = \mathcal{H}$. In order to have a model where $\Gamma(\pi) \subset \mathcal{H}$ we have to get a more detailed structure, namely the **half-side modular inclusion**. So for any $x \in X$ let $\langle \mathfrak{N}_x \subset \mathfrak{E}_x, \Omega_x \rangle$ be a $hsmi^+$ and V_x the Wiesbrock one-parameter semigroup of unitarities associated to it so $V_x \in Hstr(\mathfrak{E}_x)^+$ such that $\mathfrak{N}_x = Ad(V_x(1))\mathfrak{E}_x$. Therefore what we are interested in is that for all $t \in \mathbb{R}^+$

$$(3.3.25) \quad \begin{cases} Ad(V_x(t))(\mathfrak{E}_x) \subseteq \mathfrak{E}_x, \\ Ad(V_x(t))(\mathfrak{N}_x) \subseteq \mathfrak{N}_x. \end{cases}$$

By using the first inclusion in (3.3.25) we can set

$$\begin{cases} \mathcal{A}_V \doteq \{V_t : X \ni x \mapsto Ad(V_x(t)) \upharpoonright \mathfrak{E}_x \in Aut(\mathfrak{E}_x) \mid t \in \mathbb{R}\} \\ G_V \doteq \theta(\mathcal{A}_V) \upharpoonright \mathcal{H} \\ \overline{\mathcal{V}}_t \doteq \theta(V_t) \upharpoonright \mathcal{H}, t \in \mathbb{R}. \end{cases}$$

Hence for all $x \in X$ and $t \in \mathbb{R}$

$$\begin{cases} \overline{\mathcal{V}}_t(v)(x) = Ad(V_x(t))v(x) \\ \Psi_{\mathbb{R}}^{\mathcal{H}}(\overline{\mathcal{V}})(x)(t) = Ad(V_x(t)) \end{cases}$$

Therefore if we set \mathcal{D} and $\Gamma(\pi)$ such that

$$\mathcal{D} \subseteq \Gamma(\pi) \doteq \int^{\oplus} \mathfrak{N}_x d\mu(x) \subset \mathcal{H}$$

then by using the second inclusion in (3.3.25) we have

$$\boxed{\langle G_V, \mathcal{D} \rangle \subseteq \Gamma(\pi).}$$

3.3.3.4. Inner property of the Tomita-Takesaki modular group. For any semi-finite von Neumann algebra \mathfrak{N} and any $\phi \in \mathbf{N}_{\mathfrak{N}}$ faithful we have that the **Tomita-Takesaki modular group** $\sigma_{\mathfrak{N}}^{\phi}$ **is inner** (see [Tak, Thm. 3.14 Ch. VIII]) i.e. it is implemented by a strongly continuous group morphism $V : \mathbb{R} \rightarrow U(\mathfrak{N})$, where $U(\mathfrak{N}) \doteq \{U^{-1} = U^* \mid U \in \mathfrak{N}\}$, so in particular

$$(3.3.26) \quad V(\mathbb{R}) \subset \mathfrak{N}.$$

Now let $\langle H_{\phi}, \pi_{\phi}, \Omega_{\phi} \rangle$ be a cyclic representation associated to ϕ and $\mathfrak{N}_{\phi} \doteq \pi(\mathfrak{N}_{\phi})$ which is a von Neumann algebra ϕ being normal, then by (3.3.26) immediatedly we have

$$(3.3.27) \quad \boxed{\pi_{\phi}(V(\mathbb{R})) \subset \mathfrak{N}_{\phi}.}$$

By the invariance $\phi = \phi \circ \sigma_{\mathfrak{N}}^{\phi}$, and the cited unitary implementation we obtain that there exists W_{ϕ} unitary action on H_{ϕ} such that

$$(3.3.28) \quad \boxed{\begin{cases} Ad(W_{\phi}(t)) \circ \pi_{\phi} = Ad(\pi_{\phi}(V(t))) \circ \pi_{\phi}, \\ W_{\phi}(t) = \Delta_{\phi}^{it}, \end{cases}}$$

where the second sentence comes by [Tak, Thm, 1.2 Ch. VIII], with Δ_{ϕ} the modular operator associated to $\langle \mathfrak{N}_{\phi}, \Omega_{\phi} \rangle$.

CHAPTER 4

Section of Projections

In sections 4.1 and 4.2, except when explicitly stated, we shall maintain Notations 3.3.3.

4.1. $\langle \nu, \eta, E, Z, T \rangle$ invariant set with respect to \mathcal{F}

DEFINITION 4.1.1. *Let Z, T be two locally compact spaces, $E \in Hlcs$, $\nu \in Rad(Z)$ and $\eta \in Rad(T)^Z$. Set*

$$\mathfrak{L}_{(1,1)}(T, E, \eta, \nu) \doteq \left\{ \overline{F} \in \bigcap_{\lambda \in Z} \mathfrak{L}_1(T, E, \eta_\lambda) \mid \left(Z \ni \lambda \mapsto \int \overline{F}(s) d\eta_\lambda(s) \in E \right) \in \mathfrak{L}_1(Z, E, \nu) \right\}$$

COROLLARY 4.1.2. *Let Z be a locally compact space, $\nu \in Rad(Z)$ and $\eta \in Rad(Y)^Z$ finally let $D \in \prod_{x \in X} 2^{\mathfrak{E}_x}$ and assume (A) of Lemma 3.3.25. Thus $(\forall \overline{F} \in \mathfrak{L}_{(1,1)}(Y, \mathfrak{G}, \eta, \nu))(\forall x \in X)(\forall v \in \prod_{y \in X} D(y))$*

$$\Pr_x \circ \left[\int \left(\int \overline{F}(s) d\eta_\lambda(s) \right) d\nu(\lambda) \right] (v) = \left[\int \left(\int \Pr_x(\Psi(\overline{F}))(s) v(x) d\eta_\lambda(s) \right) d\nu(\lambda) \right].$$

PROOF. Let $\overline{F} \in \mathfrak{L}_{(1,1)}(Y, \mathfrak{G}, \eta, \nu)$, $x \in X$ and $v \in \prod_{y \in X} D(y)$. By Theorem 3.3.26

$$\Pr_x \circ \left[\int \left(\int \overline{F}(s) d\eta_\lambda(s) \right) d\nu(\lambda) \right] (v) = \int \Pr_x \circ \Psi \left(\int \overline{F}(s) d\eta_{(\cdot)}(s) \right) (\lambda)(v(x)) d\nu(\lambda).$$

Moreover $\forall \lambda \in Z$

$$\begin{aligned} \Pr_x \circ \Psi \left(\int \overline{F}(s) d\eta_{(\cdot)}(s) \right) (\lambda)(v(x)) &= \Pr_x \circ \left(\int \overline{F}(s) d\eta_\lambda(s) \right) \circ \iota_x(v(x)) \\ &= \int \Pr_x(\Psi(\overline{F}))(s) \circ \Pr_x \circ \iota_x(v(x)) d\eta_\lambda(s) \\ &= \int \Pr_x(\Psi(\overline{F}))(s) v(x) d\eta_\lambda(s), \end{aligned}$$

where in the first equality we used Prop. 3.3.19, while in the second one Theorem 3.3.26. Then the statement follows. \square

DEFINITION 4.1.3. *V is a $\langle \nu, \eta, E, Z, T \rangle$ invariant set with respect to \mathcal{F} if*

- (1) T, Z are two locally compact spaces;
- (2) $E \in Hlcs$ and $V \subseteq E$;
- (3) $\nu \in Rad(Z)$ and $\eta \in Rad(T)^Z$;
- (4) $\mathcal{F} \subseteq \mathfrak{L}_{(1,1)}(T, E, \eta, \nu)$
- (5) $\forall \overline{F} \in \mathcal{F}$

$$\left[\int \left(\int \overline{F}(s) d\eta_\lambda(s) \right) d\nu(\lambda) \right] V \subseteq V.$$

PROPOSITION 4.1.4. *Let us assume the hypotheses of Corollary 4.1.2 and V be a $\langle \nu, \eta, \mathfrak{G}, Z, Y \rangle$ invariant set with respect to \mathcal{F} such that $V \cap \prod_{y \in X} D(y) \neq \emptyset$. Then $\forall v \in V \cap \prod_{y \in X} D(y)$ and $\forall \overline{F} \in \mathcal{F}$*

$$\left(X \ni x \mapsto \left[\int \left(\int \text{Pr}_x(\Psi(\overline{F}))(s) v(x) d\eta_\lambda(s) \right) d\nu(\lambda) \right] \in \mathfrak{E}_x \right) \in V.$$

PROOF. By Corollary 4.1.2. □

4.2. Construction of classes $\Delta_\Theta \langle \mathfrak{W}, \mathfrak{D}, \mathfrak{W}, \mathcal{E}, X, \mathbb{R}^+ \rangle$ through invariant sets

In this Section we shall assume that $\mathbf{E} \doteq \{\mathfrak{E}_x\}_{x \in X}$ is a class of complex Banach spaces. Moreover $Cld(\mathfrak{E}_x)$ denotes, for all $x \in X$, the class of all closed densely defined linear operators $T_x : Dom(T_x) \subseteq \mathfrak{E}_x \rightarrow \mathfrak{E}_x$.

DEFINITION 4.2.1 (Ch. 9, §1, $n^\circ 4$ of [Kat]). *Let $M > 1$, $\beta \in \mathbb{R}$. Set $\mathcal{G}(M, \beta, \mathbf{E})$ the class of all $T \in \prod_{x \in X} Cld(\mathfrak{E}_x)$ such that $]\beta, \infty[\subseteq P(-T(x))$ ¹ and $(\forall \xi > \beta)(\forall k \in \mathbb{N})(\forall x \in X)$*

$$\|(T(x) + \xi)^{-k}\|_{B(\mathfrak{E}_x)} \leq M(\xi - \beta)^{-k}.$$

Moreover let us denote by $\{e^{-tT(x)}\}_{t \in \mathbb{R}^+}$ the strongly continuous semigroup generated by $-T(x)$.

DEFINITION 4.2.2 (Separation of the Spectrum). *See [Kat, $n^\circ 4, \S 6$, Ch. 3]. Let $M > 1$, $\beta \in \mathbb{R}$. We say that $T \in \mathcal{G}(M, \beta, \mathbf{E})$ satisfies the property of separation of the spectrum if $(\exists \Gamma)(\forall x \in X)(\exists \Sigma'_{T(x)} \subseteq \Sigma(T(x)))(\exists A_{T(x)} \in Op(\mathbb{C}))$ such that Γ is a regular closed curve in \mathbb{C} , $\Sigma'_{T(x)}$ is bounded and*

$$\Sigma'_{T(x)} \subset A_{T(x)} \subset \mathcal{O}_i(\Gamma), \Sigma''_{T(x)} \subset \mathcal{O}_e(\Gamma).$$

Here $\mathcal{O}_i(\Gamma)$ is the interior of Γ , namely the compact set of \mathbb{C} whose frontier is Γ , $\mathcal{O}_e(\Gamma) \doteq \mathbb{C} \setminus \mathcal{O}_i(\Gamma)$ is the exterior of Γ , $\Sigma(T(x))$ is the spectrum of $T(x)$, finally $\Sigma''_{T(x)} \doteq \Sigma(T(x)) \cap \mathbb{C} \setminus \Sigma'_{T(x)}$.

¹ Equivalently $-\beta, \infty[\subseteq P(T(x))$, where $P(T(x))$ is the resolvent set of any closed operator T_x .

Let $T \in \mathcal{G}(M, \beta, \mathbf{E})$ satisfy the property of separation of the spectrum, then $\forall x \in X$ we set

$$(4.2.1) \quad P(x) \doteq -\frac{1}{2\pi i} \int_{\Gamma} R(T(x); \zeta) d\zeta \in B(\mathfrak{E}_x).$$

Moreover set $R_T^\rho \in \prod_{x \in X} \mathcal{L}(\mathfrak{E}_x)^{\mathbb{R}^+}$ such that $R_T^\rho(x)(s) \doteq R(T(x); \rho(s))$, for all $x \in X$ and $s \in K(\Gamma)$, while $R_T^\rho(x)(s) \doteq \mathbf{0}$, if $s \in \mathbb{R}^+ - K(\Gamma)$. Here $R(T(x); \cdot) : P(T(x)) \ni \zeta \mapsto (T(x) - \zeta)^{-1} \in B(\mathfrak{E}_x)$ is the resolvent map of $T(x)$ and $P(T(x))$ is its resolvent set, while the integration is with respect to the norm topology on $B(\mathfrak{E}_x)$.

REMARK 4.2.3. Let $M > 1$, $\beta \in \mathbb{R}$ and $T \in \mathcal{G}(M, \beta, \mathbf{E})$ satisfy the property of separation of the spectrum. Then for all $x \in X$ by [Kat, Th. 6.17., Ch. 3], $P(x) \in \text{Pr}(\mathfrak{E}_x)$ and $\mathfrak{E}_x = M'_x \oplus M''_x$ direct sum of two closed subspaces of \mathfrak{E}_x , where $M'_x = P(x)\mathfrak{E}_x$ and $M''_x = (\mathbf{1}_x - P(x))\mathfrak{E}_x$. Moreover $T(x)$ decomposes according the previous decomposition, namely $T(x) \upharpoonright M'_x \in B(M'_x)$ such that $\Sigma(T_x \upharpoonright M'_x) = \Sigma'_{T_x}$ and $T_x \upharpoonright M''_x$ is a closed operator in M''_x such that $\Sigma(T_x \upharpoonright M''_x) = \Sigma''_{T_x}$.

DEFINITION 4.2.4. Let $K(\Gamma) \subset \mathbb{R}^+$ a compact set, A an open neighbourhood of $K(\Gamma)$ and $\rho : A \rightarrow \mathbb{C}$ be such that $\rho \in C_1(A, \mathbb{R}^2)$ ² and $\rho(K(\Gamma)) = \Gamma$. Set $\forall s \in K(\Gamma)$, $\eta_s \in \text{Radon}(\mathbb{R}^+)$ such that

$$\eta_s : \mathcal{C}_{cs}(\mathbb{R}^+) \ni f \mapsto \int_{\mathbb{R}^+} e^{\rho(s)t} f(t) dt.$$

Moreover let $\nu \in \text{Radon}(\mathbb{R}^+)$ be the $\mathbf{0}$ -extension of $\nu_0 \in \text{Radon}(K(\Gamma))$ such that

$$\nu_0 : \mathcal{C}_{cs}(K(\Gamma)) \ni g \mapsto \int_{K(\Gamma)} \frac{-g(s)}{2\pi i} \frac{d\rho}{ds}(s) ds.$$

Finally let $M > 1$, $\beta \in \mathbb{R}$ and $T \in \mathcal{G}(M, \beta, \mathbf{E})$, then we set $\mathcal{W}_T \in \prod_{x \in X} \mathbf{U}(B_s(\mathfrak{E}_x))$ such that $(\forall x \in X)(\forall t \in \mathbb{R}^+)$

$$\begin{cases} \mathcal{W}_T(x)(t) \doteq e^{-T(x)t}, \\ \overline{F}_T \doteq \Lambda(\mathcal{W}_T), \end{cases}$$

where Λ has been defined in Def. 3.3.18.

LEMMA 4.2.5. Let $M > 1$, $\beta \in \mathbb{R}$ and $T \in \mathcal{G}(M, \beta, \mathbf{E})$ satisfy the property of separation of the spectrum. Assume that there exists a closed curve Γ of which in Definition

² By identifying \mathbb{C} with \mathbb{R}^2 , so ρ is derivable with continuous derivative

4.2.2 such that

$$(4.2.2) \quad \operatorname{Re}(\Gamma) \subseteq \mathbb{R}^-.$$

Then $\forall x \in X$ and $\forall v_x \in \mathfrak{E}_x$

$$(4.2.3) \quad P(x)v_x = -\frac{1}{2\pi i} \int_{K(\Gamma)} \frac{d\rho}{ds}(s) R(T(x); \rho(s)) v_x ds,$$

and $\forall s \in K(\Gamma)$,

$$(4.2.4) \quad R(T(x), \rho(s))v_x = \int_0^\infty e^{\rho(s)t} e^{-tT(x)} v_x dt = \int_{\mathbb{R}^+} \mathcal{W}_T(x)(t) v_x d\eta_s(t).$$

Here the integration is with respect to the norm topology on \mathfrak{E}_x . Moreover let \mathfrak{G} be that in Def. 3.3.12 relative to \mathbf{E} . If $\overline{F}_T \in \mathfrak{L}_{(1,1)}(\mathbb{R}^+, \mathfrak{G}, \eta, \nu)$ and V is a $\langle \nu, \eta, \mathfrak{G}, K(\Gamma), \mathbb{R}^+ \rangle$ invariant set with respect to $\{\overline{F}_T\}$, then ³

$$(4.2.5) \quad P \bullet V \subseteq V$$

PROOF. By (4.2.1), [INT, IV.35 Th. 1], and by the norm continuity of the map $B(\mathfrak{E}_x) \ni A \mapsto Aw \in \mathfrak{E}_x$ for any $w \in \mathfrak{E}_x$, we have (4.2.3). Moreover by (4.2.2) we can apply [Kat, eq. 1.28, n°3, §1, Ch. 9] and (4.2.4) follows by Def. 4.2.4. Fix $v \in V$ so $\forall x \in X$

$$(4.2.6) \quad \begin{aligned} P(x)v(x) &= -\frac{1}{2\pi i} \int_{K(\Gamma)} \frac{d\rho}{ds}(s) R(T(x); \rho(s)) v(x) ds \\ &= -\frac{1}{2\pi i} \int_{K(\Gamma)} \frac{d\rho}{ds}(s) \left(\int_{\mathbb{R}} \mathcal{W}_T(x)(t) v(x) d\eta_s(t) \right) ds \\ &= \int_{K(\Gamma)} \left(\int_{\mathbb{R}} \operatorname{Pr}_x(\Psi(\overline{F}_T))(t) v(x) d\eta_s(t) \right) d\nu(s). \end{aligned}$$

Here the first equality comes by (4.2.3), the second one by (4.2.4) and the third one by Prop. 3.3.19 and Def. 4.2.4. Finally with the notations in Corollary 3.3.29 we have that for the strong operator topology we can choose $(\forall x \in X)(S_x = \mathcal{P}_\omega(\mathfrak{E}_x))$, thus $D(x) = \mathfrak{E}_x$, for all $x \in X$, and by Corollary 3.3.29 holds (A) of Lemma 3.3.25. Therefore the statement follows by (4.2.6) and Proposition 4.1.4. \square

COROLLARY 4.2.6. *Let us assume the hypotheses and notations of the Main Theorem 3.1.16, and the hypotheses of Lemma 4.2.5 where T is such that $-T(x)$ is the infinitesimal generator of $\mathcal{U}(x)$, for all $x \in X$. Moreover let $V \subset \prod_{x \in X} \mathfrak{E}_x$. Finally let*

³See Def. 2.2.1 for (\bullet) .

$\mathfrak{D} \doteq \langle \langle \mathfrak{B}, \gamma \rangle, \eta, X, \mathfrak{L} \rangle$ and $\langle \mathfrak{V}, \mathfrak{D}, X, \{pt\} \rangle$ be a (Θ, \mathcal{E}) –structure such that

$$(4.2.7) \quad \Gamma(\eta) \subseteq \left\{ F \in \prod_{x \in X} \mathcal{L}(\mathfrak{E}_x) \mid F \bullet V \subseteq V \right\},$$

If $\overline{F}_T \in \mathfrak{L}_{(1,1)}(\mathbb{R}^+, \mathfrak{G}, \eta, \nu)$ and V is a $\langle \nu, \eta, \mathfrak{G}, K(\Gamma), \mathbb{R}^+ \rangle$ invariant set with respect to $\{\overline{F}_T\}$, then $\{\mathcal{U}\} \in \Delta_\Theta \langle \mathfrak{V}, \mathfrak{D}, \mathfrak{W}, \mathcal{E}, X, \mathbb{R}^+ \rangle$.

Notice that $\mathcal{E}(\Theta) = \mathcal{E}$ so if $\mathcal{E} \subseteq V \subseteq \Gamma(\pi)$ then (4.2.7) implies that the (Θ, \mathcal{E}) –structure $\langle \mathfrak{V}, \mathfrak{D}, X, \{pt\} \rangle$ is compatible.

PROOF. By (3.1.20) and by (4.2.5) and (4.2.7). \square

REMARK 4.2.7. Our aim now is to see when V is a $\langle \nu, \eta, \mathfrak{G}, K(\Gamma), \mathbb{R}^+ \rangle$ invariant set with respect to $\{\overline{F}_T\}$, by maintaining the positions $\mathcal{E} \subseteq V \subseteq \Gamma(\pi)$ and (4.2.7), ensuring as remarked, that $\langle \mathfrak{V}, \mathfrak{D}, X, \{pt\} \rangle$ is compatible. By using Corollary 1.2.10, we can try to show that if V is a $\langle \nu, \eta, \mathfrak{G}, K(\Gamma), \mathbb{R}^+ \rangle$ invariant set with respect to $\Lambda(\Gamma(\rho))$ then V is a $\langle \nu, \eta, \mathfrak{G}, K(\Gamma), \mathbb{R}^+ \rangle$ invariant set with respect to $\Lambda(\Gamma^{x_\infty}(\rho))$, hence by (3.1.20), with respect to $\{\overline{F}_T\}$. Might be at first glance not a weakening, but it is indeed the most powerfull way for constructing bundles of Ω –spaces, namely that described in Def. 1.2.5, allows us to choose, when X is compact, the set $\Gamma(\rho)$, see Remark 1.2.6. Assume the notations in Def. 4.1.3. Another way, maybe better than the previous one, is the following. Let $v \in \prod_{x \in X} \mathfrak{E}_x$ and A be a subalgebra of \mathfrak{G} , thus if we set

$$V \doteq \overline{\text{span}}(Av),$$

closure in $\prod_{x \in X} \mathfrak{E}_x$, then by Lemma 3.3.14 we obtain

$$AV \subseteq V.$$

Thus if $\langle \mathcal{L}(\mathfrak{E}_x), \tau_x \rangle$ is a topological algebra for all $x \in X$, it is sufficient to take for example the closure in \mathfrak{G} of the algebra A_0 ⁴ generated by any subset of \mathfrak{G} which contains the set

$$\left\{ \int \left(\int \overline{F}(s) d\eta_\lambda(s) \right) d\nu(\lambda) \mid \overline{F} \in \mathcal{F} \right\}.$$

Notice that by the fact that $P(x) \in \text{Pr}(\mathfrak{E}_x)$ for all $x \in X$, we deduce by (4.2.6) and Corollary 4.1.2 that

$$(4.2.8) \quad a_T \doteq \int \left(\int \overline{F}_T(s) d\eta_\lambda(s) \right) d\nu(\lambda) \in \text{Pr} \left(\prod_{x \in X} \mathfrak{E}_x \right).$$

⁴which is again an algebra by Lemma 3.3.17

Thus the algebra A_0 should be generated by any subset of \mathfrak{G} which contains not only the operator a_T , otherwise $A_0 = \mathbb{K} \cdot a_T$ and then $A = \mathbb{K} \cdot a_T$ which is not interesting. The last equality comes by the fact that $\mathbb{K} \cdot v$ is a closed set for any $v \in Z - \{0\}$ where Z is a topological vector space such that Z^* separates the points of Z , for example any Hlcs. Indeed let $v, w \in Z$ such that $v \neq 0$ and $\{\lambda_\alpha\}_{\alpha \in D}$ a net in \mathbb{K} such that $\lim_{\alpha \in D} \lambda_\alpha v = w$. Thus there exists $\phi \in Z^*$ such that $\phi(v) \neq 0$ and $\{\lambda_\alpha \phi(v)\}_{\alpha \in D}$ is a Cauchy net in \mathbb{K} . But $\phi(v) \neq 0$ so also $\{\lambda_\alpha\}_{\alpha \in D}$ is a Cauchy net in \mathbb{K} , let $\mu = \lim_{\alpha \in D} \lambda_\alpha$. Thus $w = \lim_{\alpha \in D} \lambda_\alpha v = \mu v$, which show that $\mathbb{K}v$ is closed in Z .

4.3. Construction of classes $\Delta_\Theta \langle \mathfrak{V}, \mathfrak{D}, \mathfrak{W}, \mathcal{E}, X, \mathbb{R}^+ \rangle$ through the generalized Lebesgue Theorem

ASSUMPTIONS 4.3.1. In this section X is a topological space, Y is a locally compact space μ is a Radon measure on Y , and $\mathfrak{V} = \langle \langle \mathfrak{E}, \tau \rangle, \pi, X, \mathfrak{N} \rangle$ is a bundle of Ω -spaces, we indicate with $\mathfrak{N} \doteq \{\nu_j \mid j \in J\}$ the directed set of seminorms on \mathfrak{E} .

DEFINITION 4.3.2. Let $Z \in \text{Hlcs}$ and $\{\psi_i \mid i \in I\}$ a fundamental set of seminorms on Z . We denote by

$$(Z^Y)_s$$

the Hlcs whose linear space support is Z^Y and whose Hlct is that generated by the following set of seminorms

$$(4.3.1) \quad \begin{cases} \{q_s^i \mid s \in Y, i \in I\}, \\ q_s^i : Z^Y \ni f \mapsto \psi_i(f(s)). \end{cases}$$

Moreover for any $B \subseteq Z^Y$ we shall denote by B_s the Hlc subspace of $(Z^Y)_s$. Notice that this definition is well-set being independent by the choice of the fundamental set of seminorms, indeed the topology is that of uniform convergence over the finite subsets of Y .

DEFINITION 4.3.3. Set

$$\begin{cases} \overset{\mu}{\blacklozenge} : \prod_{x \in X} \mathfrak{L}_1(Y, \mathfrak{E}_x, \mu) \rightarrow \prod_{x \in X} \mathfrak{E}_x, \\ \overset{\mu}{\blacklozenge}(H)(x) \doteq \int H(x)(s) d\mu(s) \in \mathfrak{E}_x, \end{cases}$$

for all $H \in \prod_{x \in X} \mathfrak{L}_1(Y, \mathfrak{E}_x, \mu)$ and for all $x \in X$

DEFINITION 4.3.4. *Set*

$$\begin{cases} \star : \prod_{x \in X} \mathcal{L}(\mathfrak{E}_x)^Y \times \prod_{x \in X} \mathfrak{E}_x \rightarrow \prod_{x \in X} \mathfrak{E}_x^Y, \\ (\forall x \in X)(\forall s \in Y)(F \star v)(x)(s) \doteq F(x)(s)(v(x)). \end{cases}$$

DEFINITION 4.3.5. $\langle \mathfrak{V}, \mathfrak{Z} \rangle$ are μ -related if

- (1) $\mathfrak{Z} \doteq \langle \langle \mathfrak{T}, \gamma \rangle, \zeta, X, \mathfrak{K} \rangle$ be a bundle of Ω -spaces;
- (2) for all $x \in X$ ⁵

$$(4.3.2) \quad \begin{cases} \mathfrak{T}_x \subseteq Meas(Y, \mathfrak{E}_x, \mu) \cap \mathfrak{L}_1(Y, \mathfrak{E}_x, \mu)_s, \\ \mathfrak{K}_x = \left\{ \sup_{(s,j) \in O} q_{(s,j)}^x \mid O \in \mathcal{P}_\omega(Y \times J) \right\}, \\ q_{(s,j)}^x : \mathfrak{T}_x \ni f_x \mapsto \nu_j(f_x(s)), \forall s \in Y, j \in J; \end{cases}$$

$$(3) \quad \Gamma(\zeta) \subset \left[\prod_{x \in X}^b \mathfrak{T}_x \right]_{ui};$$

$$(4) \quad \overset{\mu}{\blacklozenge}(\Gamma(\zeta)) \subseteq \Gamma(\pi).$$

Here we set for all $\mathcal{A} \subseteq \prod_{x \in X}^b \mathfrak{T}_x$

$$(4.3.3) \quad [\mathcal{A}]_{ui} \doteq \left\{ H \in \mathcal{A} \mid (\forall j \in J) \left(\int_Y^{\bullet} \sup_{x \in X} \nu_j(H(x)(s)) d|\mu|(s) < \infty \right) \right\}$$

Finally $\langle \mathfrak{V}, \mathfrak{Z}, \mathbf{H} \rangle$ are μ -related if

- (1) $\mathbf{H} = \{\mathbf{H}_x\}_{x \in X}$ such that $\mathbf{H}_x \subseteq \mathcal{L}(\mathfrak{E}_x)^Y$ for all $x \in X$,
- (2) $\langle \mathfrak{V}, \mathfrak{Z} \rangle$ are μ -related
- (3)

$$(4.3.4) \quad \left(\prod_{x \in X} \mathbf{H}_x \right) \star \left(\prod_{x \in X} \mathfrak{E}_x \right) \subseteq \prod_{x \in X} \mathfrak{T}_x.$$

THEOREM 4.3.6 (GLT). *Let $\langle \mathfrak{V}, \mathfrak{Z} \rangle$ be μ -related. Then for all $x \in X$*

$$\overset{\mu}{\blacklozenge}([\Gamma_\diamond^x(\zeta)]_{ui}) \subseteq \Gamma_\diamond^x(\pi).$$

PROOF. Let $x \in X$ and $F \in [\Gamma_\diamond^x(\zeta)]_{ui}$ thus by Corollary 1.2.10 there exists $\eta \in \Gamma(\zeta)$ such that for all $j \in J, s \in Y$

$$(4.3.5) \quad \begin{cases} F(x) = \eta(x) \\ \lim_{z \rightarrow x} \nu_j(F(z)(s) - \eta(z)(s)) = 0. \end{cases}$$

⁵ In case \mathfrak{E}_x is a Banach space $Meas(Y, \mathfrak{E}_x, \mu) \cap \mathfrak{L}_1(Y, \mathfrak{E}_x, \mu)_s = \mathfrak{L}_1(Y, \mathfrak{E}_x, \mu)_s$.

Fix $j \in J$ thus by [INT, Prop.6, No2, §1, Ch. 6] for all $z \in X$

$$(4.3.6) \quad \nu_j \left(\int (F(z)(s) - \eta(z)(s)) d\mu(s) \right) \leq \int^\bullet \nu_j(F(z)(s) - \eta(z)(s)) d|\mu|(s)$$

Moreover ν_j^z is continuous by definition of bundles of Ω -spaces, while $F(z)$ and $\eta(z)$ are by construction μ -measurable, hence by [INT, Thm. 1; Cor. 3, n°3, §5, Ch. 4] the map $Y \ni s \mapsto \nu_j(F(z)(s) - \eta(z)(s))$ is μ -measurable thus $|\mu|$ -measurable. Moreover by the hypothesis on F and by (3) of Def. 4.3.5

$$(4.3.7) \quad \int^\bullet \nu_j(F(z)(s) - \eta(z)(s)) d|\mu|(s) \leq \int^\bullet \left(\sup_{x \in X} \nu_j(F(x)(s)) + \sup_{x \in X} \nu_j(\eta(x)(s)) \right) d|\mu|(s) < \infty.$$

Therefore by [INT, Prp..9, No3, §1, Ch. 5] the map $Y \ni s \mapsto \nu_j(F(z)(s) - \eta(z)(s))$ is $|\mu|$ -essentially integrable hence by the fact that $\int_Y^\bullet f d|\mu| = \int_Y f d|\mu|$ for all $|\mu|$ -essentially integrable map f , we have by (4.3.6)

$$(4.3.8) \quad \nu_j \left(\int (F(z)(s) - \eta(z)(s)) d\mu(s) \right) \leq \int \nu_j(F(z)(s) - \eta(z)(s)) d|\mu|(s).$$

Let $\{z_n\}_n \subset X$ be such that $\lim_{n \in \mathbb{N}} z_n = x$ thus by (4.3.5)

$$(4.3.9) \quad \lim_{n \in \mathbb{N}} \nu_j(F(z_n)(s) - \eta(z_n)(s)) = 0,$$

For all $s \in Y$

$$\nu_j(F(z)(s) - \eta(z)(s)) \leq \sup_{x \in X} \nu_j(F(x)(s)) + \sup_{x \in X} \nu_j(\eta(x)(s))$$

thus by the hypothesis on F , by (3) of Def. 4.3.5, by the fact that $\int_Y^\bullet \leq \int_Y^*$, by (4.3.9), and by the Lebesgue Theorem [INT, Th.6, No7, §3, Ch. 4] we have

$$(4.3.10) \quad \lim_{n \in \mathbb{N}} \int \nu_j(F(z_n)(s) - \eta(z_n)(s)) d|\mu|(s) = 0.$$

Finally by (4.3.8), (4.3.10) and the fact that X is metrizable we obtain

$$\lim_{z \rightarrow x} \nu_j \left(\int F(z)(s) d\mu(s) - \int \eta(z)(s) d\mu(s) \right) = 0,$$

thus the statement follows by hypothesis (4) and Corollary 1.2.10. \square

DEFINITION 4.3.7 ($(\Theta, \mathcal{E}, \mu)$ -structures). We say that $\langle \mathfrak{V}, \mathfrak{Q}, X, Y \rangle$ is a $(\Theta, \mathcal{E}, \mu)$ -structure if

- (1) $\mathfrak{V} \doteq \langle \langle \mathfrak{E}, \tau \rangle, \pi, X, \mathfrak{N} \rangle$ is a bundle of Ω -spaces;
- (2) $\mathcal{E} \subseteq \Gamma(\pi)$;

- (3) $\Theta \subseteq \prod_{x \in X} \text{Bounded}(\mathfrak{E}_x)$;
- (4) $\forall B \in \Theta$
 - (a) $\mathbf{D}(B, \mathcal{E}) \neq \emptyset$;
 - (b) $\bigcup_{B \in \Theta} \mathcal{B}_B^x$ is total in \mathfrak{E}_x for all $x \in X$;
- (5) $\Omega \doteq \langle \langle \mathfrak{H}, \gamma \rangle, \xi, X, \mathfrak{Y} \rangle$ is a bundle of Ω -spaces such that for all $x \in X$

$$(4.3.11) \quad \left\{ \begin{array}{l} \mathfrak{H}_x \subseteq \mathcal{L}_1(Y, \mathcal{L}_{S_x}(\mathfrak{E}_x), \mu)_s, \\ \mathfrak{Y}_x = \left\{ \sup_{(t,j,B) \in O} P_{(t,j,B)}^x \mid O \in \mathcal{P}_\omega(Y \times J \times \Theta) \right\} \\ P_{(t,j,B)}^x : \mathfrak{H}_x \ni F \mapsto \sup_{v \in \mathbf{D}(B, \mathcal{E})} \nu_j(F(t)v(x)), \forall t \in Y, B \in \Theta, j \in J. \end{array} \right.$$

Here S_x , \mathcal{B}_B^x and $\mathbf{D}(B, \mathcal{E})$ are defined in (2.2.1). Moreover $\langle \mathfrak{Y}, \Omega, X, Y \rangle$ is an invariant $(\Theta, \mathcal{E}, \mu)$ -structure if it is a $(\Theta, \mathcal{E}, \mu)$ -structure such that

$$(4.3.12) \quad \left\{ F \in \prod_{z \in X}^b \mathfrak{H}_z \mid (\forall t \in Y)(F_t \bullet \mathcal{E}(\Theta) \subseteq \Gamma(\pi)) \right\} = \Gamma(\xi).$$

DEFINITION 4.3.8. Let μ_λ for all $\lambda > 0$ be defined as in Def. 3.1.8, let $\Omega = \langle \langle \mathfrak{H}, \gamma \rangle, \xi, X, \mathfrak{S} \rangle$, $\langle \mathfrak{Y}, \Omega, X, \mathbb{R}^+ \rangle$ be a $(\Theta, \mathcal{E}, \mu)$ -structure, $x \in X$, $\mathcal{O} \subseteq \Gamma(\xi)$. and $\mathcal{D} \subseteq \Gamma(\pi)$. Set

$$\mathbf{Lap}(\mathfrak{Y})(x) \doteq \bigcap_{\lambda > 0} \mathcal{L}_1(\mathbb{R}^+, \mathcal{L}_{S_x}(\mathfrak{E}_x); \mu_\lambda).$$

Assume that

$$(4.3.13) \quad \Gamma_{\mathcal{O}}^x(\xi) \bigcap \mathbf{Lap}(\mathfrak{Y})(x) \neq \emptyset$$

We say that $\langle \mathfrak{Y}, \Omega, X, \mathbb{R}^+ \rangle$ has the weak-Laplace duality property on \mathcal{O} and \mathcal{D} at x , shortly $w - \mathbf{LD}_x(\mathcal{O}, \mathcal{D})$ if $\forall \lambda > 0$

$$\blacksquare_{\mu_\lambda} \left(\Gamma_{\mathcal{O}}^x(\xi) \bigcap \mathbf{Lap}(\mathfrak{Y})(x), \Gamma_{\mathcal{D}}^x(\pi) \right) \subseteq \Gamma^x(\pi).$$

DEFINITION 4.3.9. Let $\langle \mathfrak{Y}, \mathfrak{W}, X, Y \rangle$ be a (Θ, \mathcal{E}) -structure and denote $\mathfrak{W} \doteq \langle \langle \mathfrak{M}, \gamma \rangle, \rho, X, \mathfrak{R} \rangle$. Assume that for all $x \in X$

$$(4.3.14) \quad \mathfrak{M}_x \subseteq \mathcal{L}_1(Y, \mathcal{L}_{S_x}(\mathfrak{E}_x), \mu).$$

Set $\mathbf{M}^\mu \doteq \{ \langle \mathbf{M}_x^\mu, \mathfrak{Y}_x \rangle \}_{x \in X}$ where for all $x \in X$

$$\left\{ \begin{array}{l} \mathbf{M}_x^\mu \doteq \overline{\{ \sigma(x) \mid \sigma \in \Gamma(\rho) \}}, \\ \mathfrak{Y}_x \doteq \{ \sup_{(t,j,B) \in O} P_{(t,j,B)}^x \mid O \in \mathcal{P}_\omega(Y \times J \times \Theta) \}, \end{array} \right.$$

where the closure is in the space

$$\mathfrak{L}_1(Y, \mathcal{L}_{S_x}(\mathfrak{E}_x), \mu)_s,$$

and \mathbf{M}_x^μ has to be considered as Hlc subspace of it, finally here $P_{(t,j,B)}^x$ is defined on \mathbf{M}_x^μ as in (4.3.11). Notice that \mathbf{M}^μ is a nice family of Hlcs, and that $\Gamma(\rho)$ satisfies by construction $FM(3)$ with respect to \mathbf{M}^μ . Moreover by (2.2.5) and the fact that $\{t\} \in \text{Comp}(Y)$ for all $t \in Y$

$$(4.3.15) \quad P_{(t,j,B)}^x = q_{(\{t\},j,B)}^x.$$

By [Gie, Cor.1.6.(iii)] we deduce that $\Gamma(\rho)$ satisfies $FM(4)$ with respect to $\{\langle \mathfrak{M}_x, \mathfrak{R}_x \rangle\}_{x \in X}$. Therefore we obtain by (2.2.4) and (4.3.15) that for all $t \in Y$, $j \in J$, $B \in \Theta$ and for all $\sigma \in \Gamma(\rho)$

$$X \ni x \mapsto P_{(t,j,B)}^x(\sigma(x)) \text{ is u.s.c..}$$

Moreover the upper envelope of a finite set of u.s.c. maps is an u.s.c. map, see [GT, Thm.4,§6.2.,Ch.4], therefore for all $O \in \mathcal{P}_\omega(Y \times J \times \Theta)$

$$(4.3.16) \quad X \ni x \mapsto \sup_{(t,j,B) \in O} P_{(t,j,B)}^x(\sigma(x)) \text{ is u.s.c..}$$

Hence $\Gamma(\rho)$ satisfies $FM(4)$ with respect to \mathbf{M}^μ . Finally by the boundedness of $\Gamma(\rho)$ by definition and by (4.3.15) we have also that for all $\sigma \in \Gamma(\rho)$ and $O \in \mathcal{P}_\omega(\text{Compact}(Y) \times J \times \Theta)$

$$\sup_{x \in X} \sup_{(t,j,B) \in O} P_{(t,j,B)}^x(\sigma(x)) < \infty.$$

Therefore we can construct the bundle generated by the couple $\langle \mathbf{M}^\mu, \Gamma(\rho) \rangle$, see Def. 1.2.5

$$\mathfrak{V}(\mathbf{M}^\mu, \Gamma(\rho))$$

We shall call $\langle \mathfrak{V}, \mathfrak{V}(\mathbf{M}^\mu, \Gamma(\rho)), X, Y \rangle$ the $(\Theta, \mathcal{E}, \mu)$ – structure underlying $\langle \mathfrak{V}, \mathfrak{W}, X, Y \rangle$.

DEFINITION 4.3.10. Let $\langle \mathfrak{V}, \mathfrak{Q}, X, Y \rangle$ be a $(\Theta, \mathcal{E}, \mu)$ – structure and $A \subset \prod_{x \in X} \mathfrak{H}_x$. Define A_{peq} as the set of all pointwise equicontinuous elements in A , and A_{ceq} as the set of all compactly equicontinuous elements in A , see Def. 2.2.9.

REMARK 4.3.11. Lemma 2.2.5 holds by replacing a (Θ, \mathcal{E}) – structure $\langle \mathfrak{V}, \mathfrak{W}, X, Y \rangle$ with a $(\Theta, \mathcal{E}, \mu)$ – structure $\langle \mathfrak{V}, \mathfrak{Q}, X, Y \rangle$ and $K \in \text{Comp}(Y)$ with $t \in Y$. In what

follows when referring to Lemma 2.2.5 for a $(\Theta, \mathcal{E}, \mu)$ – structure we shall mean the corresponding result with the replacements described here.

LEMMA 4.3.12. Let $\langle \mathfrak{V}, \mathfrak{Q}, X, Y \rangle$ be a $(\Theta, \mathcal{E}, \mu)$ – structure and $\langle \mathfrak{V}, \mathfrak{Z}, \mathbf{H} \rangle$ be μ –related, where $\mathfrak{Q} \doteq \langle \langle \mathfrak{H}, \gamma \rangle, \xi, X, \mathfrak{Y} \rangle$ and $\mathbf{H}_x \doteq \mathfrak{H}_x$ for all $x \in X$. Thus

$$\Gamma(\xi) \star \mathcal{E}(\Theta) \subseteq \Gamma(\zeta) \Rightarrow (\forall x \in X) (\Gamma_{\diamond}^x(\xi)_{peq} \star \Gamma_{\mathcal{E}(\Theta)}^x(\pi) \subseteq \Gamma_{\diamond}^x(\zeta)).$$

PROOF. Let $j \in J$, $x \in X$ and $w \in \Gamma_{\mathcal{E}(\Theta)}^x(\pi)$, so there exists $v \in \mathcal{E}(\Theta)$ such that $v(x) = w(x)$ then by Cor. 1.2.10

$$(4.3.17) \quad \lim_{z \rightarrow x} \nu_j(w(z) - v(z)) = 0.$$

Moreover let $F \in \Gamma_{\diamond}^x(\xi)$, so by Lemma 2.2.5 $\exists \sigma \in \Gamma(\xi)$ such that $F(x) = \sigma(x)$ and for all $t \in Y$

$$(4.3.18) \quad \lim_{z \rightarrow x} \nu_j(F(z)(t)v(z) - \sigma(z)(t)v(z)) = 0.$$

Moreover $(\forall t \in Y)(\exists M_{(t,j)} > 0)(\exists j_1 \in J)(\forall z \in X)$

$$\begin{aligned} \nu_j((F \star w)(z)(t) - (\sigma \star v)(z)(t)) &= \nu_j(F(z)(t)w(z) - \sigma(z)(t)v(z)) \\ &\leq \nu_j(F(z)(t)(w(z) - v(z))) + \nu_j(F(z)(t)v(z) - \sigma(z)(t)v(z)) \\ &\leq M_{(t,j)}\nu_{j_1}(w(z) - v(z)) + \nu_j(F(z)(t)v(z) - \sigma(z)(t)v(z)). \end{aligned}$$

Therefore by (4.3.17) and (4.3.18) for all $t \in Y$

$$\lim_{z \rightarrow x} \nu_j((F \star w)(z)(t) - (\sigma \star v)(z)(t)) = 0.$$

Moreover $(\forall t \in Y)(\exists M_{(t,j)} > 0)(\exists j_1 \in J)$

$$(4.3.19) \quad \sup_{z \in X} \nu_j((F \star w)(z)(t)) \leq M_{(t,j)} \sup_{z \in X} \nu_{j_1}(w(z)) < \infty.$$

By the antecedent of the implication of the statement we deduce that $\sigma \star v \in \Gamma(\zeta)$ hence the statement follows by Cor. 1.2.10, (4.3.2), by the fact that by (4.3.4) $F \star w \in \prod_{x \in X} \mathfrak{T}_x$ and by (4.3.19). \square

PROPOSITION 4.3.13. Let $\langle \mathfrak{V}, \mathfrak{W}, X, Y \rangle$ be a compatible (Θ, \mathcal{E}) – structure. Then for all $x \in X$

$$(4.3.20) \quad (\Gamma_{\diamond}^x(\rho)_{peq})_t \bullet \Gamma_{\mathcal{E}(\Theta)}^x(\pi) \subseteq \Gamma_{\diamond}^x(\pi)$$

PROOF. Notice that $(F \star v)(t) = F(\cdot)(t) \bullet v$ with obvious meaning of the symbols. Thus if we set $Y = \{pt\}$ the statement follows by Lm. 4.3.12. \square

COROLLARY 4.3.14. Let $\langle \mathfrak{V}, \mathfrak{Q}, X, Y \rangle$ be a $(\Theta, \mathcal{E}, \mu)$ – structure and $\langle \mathfrak{V}, \mathfrak{Z}, \mathbf{H} \rangle$ be μ –related, Thus for all $x \in X$

$$\Gamma(\xi) \star \mathcal{E}(\Theta) \subseteq \Gamma(\zeta) \Rightarrow \diamond^\mu \left([\Gamma_\diamond^x(\xi)_{peq} \star \Gamma_{\mathcal{E}(\Theta)}^x(\pi)]_{ui} \right) \subseteq \Gamma_\diamond^x(\pi).$$

Here $\mathfrak{Q} \doteq \langle \langle \mathfrak{H}, \gamma \rangle, \xi, X, \mathfrak{Y} \rangle$, $\mathfrak{Z} \doteq \langle \langle \mathfrak{T}, \delta \rangle, \zeta, X, \mathfrak{K} \rangle$ and $\mathbf{H}_x \doteq \mathfrak{H}_x$ for all $x \in X$.

PROOF. By Theorem 4.3.6 and Lemma 4.3.12. \square

LEMMA 4.3.15. Let $\langle \mathfrak{V}, \mathfrak{Q}, X, Y \rangle$ be an invariant $(\Theta, \mathcal{E}, \mu)$ – structure where Θ defined in (3.1.16). Then for all $x \in X$

$$(4.3.21) \quad \left\{ H \in \left[\left(\prod_{z \in X}^b \mathfrak{H}_z \right)_\diamond^x \mid (\forall t \in Y)(H(\cdot)(t) \bullet \mathcal{E}(\Theta) \subseteq \Gamma^x(\pi)) \right] \right\} \subseteq \Gamma_\diamond^x(\xi).$$

PROOF. Let $v \in \mathcal{E}(\Theta)$, $t \in Y$ and H belong to the set in the left side of (4.3.21). Thus by (4.3.12) $\exists F \in \Gamma(\xi)$ such that $F_t \bullet v \in \Gamma(\pi)$, $F(x) = H(x)$ and $H(\cdot)(t) \bullet v \in \Gamma^x(\pi)$ by construction. Then by Corollary 1.2.10 we obtain for all $j \in J$

$$\lim_{z \rightarrow x} \nu_j(H(z)(t)v(z) - F(z)(t)v(z)).$$

Therefore the statement follows by Lemma 2.2.5 and (3.1.17). \square

REMARK 4.3.16. Notice that Lemma 4.3.15 holds if we replace invariant $(\Theta, \mathcal{E}, \mu)$ – structure with invariant (Θ, \mathcal{E}) – structure, see Def. 2.2.2, and assume that $\text{Comp}(Y) = \{\{t\} \mid t \in Y\}$.

COROLLARY 4.3.17. Assume that \mathfrak{V} is a Banach bundle and set $\mathbf{E} \doteq \{\mathfrak{E}_x\}_{x \in X}$. By using the notations of Definitions 4.2.4 and 4.2.2 assume that $M > 1$, $\beta \in \mathbb{R}$ and $T \in \mathcal{G}(M, \beta, \mathbf{E})$ satisfy the property of separation of the spectrum, moreover that there exists a closed curve Γ of which in Definition 4.2.2 such that

$$(4.3.22) \quad \begin{cases} \text{Re}(\Gamma) \subseteq \mathbb{R}^-, \\ \beta \geq 0 \Rightarrow -\beta \notin \text{Re}(\Gamma). \end{cases}$$

Moreover assume that for all $\mu \in \{\nu, \eta_s \mid s \in K(\Gamma)\}$

- (1) $\langle \mathfrak{V}, \mathfrak{Q}, X, \mathbb{R}^+ \rangle$ is an invariant $(\Theta, \mathcal{E}, \mu)$ – structure⁶;
- (2) $\langle \mathfrak{V}, \mathfrak{Z}, \mathbf{H} \rangle$ is μ –related;
- (3) \mathfrak{Q} is full;

⁶ See Prop. 3.3.2 or more in general [Sil, Corollary 2.6.].

(4) for all $z \in X$

$$(4.3.23) \quad \mathcal{C}_{cs}(\mathbb{R}^+, \mathcal{L}_{S_z}(\mathfrak{E}_z)) \subseteq \mathfrak{H}_z;$$

$$(5) \quad \Gamma(\xi) \star \mathcal{E}(\Theta) \subseteq \Gamma(\zeta).$$

Here Θ is defined in (3.1.16), $\mathfrak{Q} \doteq \langle \langle \mathfrak{H}, \gamma \rangle, \xi, X, \mathfrak{R} \rangle$, $\mathfrak{Z} \doteq \langle \langle \mathfrak{T}, \gamma \rangle, \zeta, X, \mathfrak{R} \rangle$ and $\mathbf{H}_x \doteq \mathfrak{H}_x$, for all $x \in X$.

Then for all $x \in X$

$$(4.3.24) \quad \mathcal{W}_T \in \Gamma^x(\xi) \Rightarrow P \bullet \Gamma_{\mathcal{E}(\Theta)}^x(\pi) \subseteq \Gamma^x(\pi).$$

Moreover if $\langle \mathfrak{V}, \mathfrak{D}, X, \{pt\} \rangle$ is an invariant (Θ, \mathcal{E}) -structure such that $\mathfrak{B}_x = \mathcal{L}(\mathfrak{E}_x)$ for all x and \mathfrak{D} is full then for all $x \in X$

$$(4.3.25) \quad \mathcal{W}_T \in \Gamma^x(\xi) \Rightarrow P \in \Gamma^x(\eta),$$

where $\mathfrak{D} \doteq \langle \langle \mathfrak{B}, \gamma \rangle, \eta, X, \mathfrak{L} \rangle$.

PROOF. In this proof we denote R_T^ρ simply by R^ρ . R^ρ is $K(\Gamma)$ -supported by construction, moreover by the analyticity of the resolvent map $R^\rho(x)$ is $\|\cdot\|_{B(\mathfrak{E}_x)}$ -continuous hence continuous as a map valued in $\mathcal{L}_{S_x}(\mathfrak{E}_x)$.⁷ So

$$R^\rho \in \prod_{z \in X} \mathcal{C}_{cs}(\mathbb{R}^+, \mathcal{L}_{S_z}(\mathfrak{E}_z)),$$

hence by (4.3.23) follows

$$(4.3.26) \quad R^\rho \in \prod_{z \in X} \mathfrak{H}_z.$$

By (4.2.4) for all $s \in K(\Gamma)$, $x \in X$ and $\forall v_x \in \mathfrak{E}_x$

$$(4.3.27) \quad \begin{aligned} \|R(T(x), \rho(s))v_x\| &\leq \int_{\mathbb{R}^+}^* e^{-|Re(\rho(s))|t} \|e^{-tT(x)}v_x\| dt \\ &\leq M \|v_x\| \int_{\mathbb{R}^+}^* e^{(\beta - |Re(\rho(s))|)t} dt = \frac{M \|v_x\|}{\beta - |Re(\rho(s))|}, \end{aligned}$$

where $\int_{\mathbb{R}^+}^*$ is the upper integral on \mathbb{R}^+ with respect to the Lebesgue measure. We considered in the first inequality [INT, Prop. 6, n° , §1, Ch. 6], in the second one the inequality [Kat, (1.37), n° 4, §1, Ch. 9], finally in the equality the Laplace transform of the map

⁷ Indeed for all E normed space the topology on $B(E)$ is stronger of that on $\mathcal{L}_S(E)$. For that it is sufficient to show $B(E) = \mathcal{L}_{Bounded(E)}(E)$. To this end note that $\sup_{v \in E_r} \|Av\| \leq r \sup_{v \in E_1} \|Av\|$ for all $r > 0$ and $A \in \mathcal{L}(E)$, where $E_r = \{v \in E \mid \|v\| \leq r\}$.

$\exp(\beta t)$. Therefore by (4.3.26) and (4.3.27)

$$R^\rho \in \left(\prod_{z \in X} \mathfrak{H}_z \right)_{peq}.$$

Thus by (4.3.27), (3.1.16) and (4.3.11)

$$(4.3.28) \quad R^\rho \in \left(\prod_{z \in X}^b \mathfrak{H}_z \right)_{peq}.$$

By (4.3.26) and (4.3.4) we have that $R^\rho \star v \in \prod_{z \in X} \mathfrak{T}_z$ for all $v \in \prod_{x \in X}^b \mathfrak{E}_x$. By hypothesis (4.3.22) we deduce that $\frac{1}{\beta - |Re(\rho(s))|}$ is defined on $K(\Gamma)$, hence continuous and integrable in it, thus by (4.3.27)

$$(4.3.29) \quad R^\rho \star v \in \left[\prod_{z \in X} \mathfrak{T}_z \right]_{ui}.$$

By the continuity of $\frac{1}{\beta - |Re(\rho(s))|}$ on $K(\Gamma)$ we deduce that the map $\frac{|\frac{d\rho}{ds}(s)|}{\beta - |Re(\rho(s))|}$ is integrable in $K(\Gamma)$. Hence by (4.3.27) and (4.2.3)

$$(4.3.30) \quad \sup_{x \in X} \|P(x)\|_{B(\mathfrak{E}_x)} \leq D \doteq \frac{1}{2\pi i} \int_{K(\Gamma)} \frac{M \left| \frac{d\rho}{ds}(s) \right|}{\beta - |Re(\rho(s))|} ds.$$

Therefore for all $v \in \mathcal{E}$ by considering that $\mathcal{E} \subset \prod_{x \in X}^b \mathfrak{E}_x$

$$\sup_{x \in X} \|P(x)v(x)\|_{B(\mathfrak{E}_x)} \leq D \sup_{x \in X} \|v(x)\|_{\mathfrak{E}_x} < \infty$$

Thus

$$(4.3.31) \quad P \in \prod_{x \in X}^b \mathfrak{B}_x.$$

Let $x \in X$ and $v \in \Gamma_{\mathcal{E}(\Theta)}^x(\pi)$. By (4.2.4) for all $s \in K(\Gamma)$

$$(4.3.32) \quad (R^\rho \star v)(x)(s) = \diamond^{\eta_s}(\mathcal{W}_T \star v)(x).$$

Moreover by (4.2.3)

$$(4.3.33) \quad P \bullet v = \diamond^\nu(R^\rho \star v).$$

Notice that $(R^\rho \star v)(x)(s) = (R^\rho(\cdot)(s) \bullet v)(x)$ so by (4.3.32) for all $s \in K(\Gamma)$

$$(4.3.34) \quad R^\rho(\cdot)(s) \bullet v = \diamond^{\eta_s}(\mathcal{W}_T \star v).$$

If $\mathcal{W}_T \in \Gamma^x(\xi)$ then by [Kat, Ch. 9, §1, $n^\circ 4$, (1.37)] and the hyp. that \mathfrak{Q} is full follows that $\mathcal{W}_T \in \Gamma_\diamond^x(\xi)_{peq}$. Therefore by Corollary 4.3.14 and (4.3.34) for all $s \in K(\Gamma)$

$$(4.3.35) \quad R^\rho(\cdot)(s) \bullet v \in \Gamma^x(\pi).$$

By construction $\mathcal{E}(\Theta) \subseteq \Gamma(\pi)$ so $\mathcal{E}(\Theta) \subseteq \Gamma_{\mathcal{E}(\Theta)}^x(\pi)$, hence

$$(4.3.36) \quad R^\rho(\cdot)(s) \bullet \mathcal{E}(\Theta) \subseteq \Gamma^x(\pi).$$

Moreover by the hypothesis that \mathfrak{Q} is full and by (4.3.28) follows that

$$R^\rho \in \left[\left(\prod_{z \in X}^b \mathfrak{H}_z \right)_{\diamond}^x \right]_{peq}.$$

Hence by Lemma 4.3.15 and (4.3.36)

$$(4.3.37) \quad R^\rho \in \Gamma_\diamond^x(\xi)_{peq}.$$

Finally (4.3.24) follows by (4.3.37), (4.3.33), (4.3.29) and Corollary 4.3.14. In conclusion by (4.3.31), (4.3.30) and the hyp. that \mathfrak{D} is full

$$P \in \left[\left(\prod_{z \in X}^b \mathfrak{B}_z \right)_{\diamond}^x \right]_{peq}.$$

Thus (4.3.25) follows by (4.3.24), Remark 4.3.16 and by $\mathcal{E}(\Theta) \subseteq \Gamma_{\mathcal{E}(\Theta)}^x(\pi)$. \square

REMARK 4.3.18. *The statements of Corollary 4.3.17 hold by replacing “for all $x \in X$ ” by “ $x_\infty \in X$ ”, and the hypothesis that $\langle \mathfrak{V}, \mathfrak{Z}, \mathbf{H} \rangle$ is μ -related and $\langle \mathfrak{V}, \mathfrak{Q}, X, \mathbb{R}^+ \rangle$ is an invariant $(\Theta, \mathcal{E}, \mu)$ -structure for all $\mu \in \{\nu, \eta_s \mid s \in K(\Gamma)\}$, with the following two*

- (1) *$\langle \mathfrak{V}, \mathfrak{Z}, \mathbf{H} \rangle$ is ν -related and $\langle \mathfrak{V}, \mathfrak{Q}, X, \mathbb{R}^+ \rangle$ is an invariant $(\Theta, \mathcal{E}, \nu)$ -structure;*
- (2) *$R^\rho(\cdot)(s) \bullet \mathcal{E} \subseteq \Gamma^{x_\infty}(\pi)$ for all $s \in K(\Gamma)$.*

Indeed (4.3.36) follows for $x = x_\infty$ and the proof runs as that of Cor. 4.3.17.

The following result shall permit to apply Corollary 4.3.17 to the Main Theorem 3.1.16.

PROPOSITION 4.3.19. *Let $\langle \mathfrak{V}, \mathfrak{W}, X, Y \rangle$ be a (Θ, \mathcal{E}) -structure and denote $\mathfrak{W} \doteq \langle \langle \mathfrak{M}, \gamma \rangle, \rho, X, \mathfrak{R} \rangle$. Assume that (4.3.14) holds for all $x \in X$ and let $\langle \mathfrak{V}, \mathfrak{V}(\mathbf{M}^\mu, \Gamma(\rho)), X, Y \rangle$ be the $(\Theta, \mathcal{E}, \mu)$ -structure underlying $\langle \mathfrak{V}, \mathfrak{W}, X, Y \rangle$. Then for all $x \in X$*

$$(4.3.38) \quad \Gamma_\diamond^x(\rho) \subseteq \Gamma_{\Gamma(\rho)}^x(\pi_{\mathbf{M}^\mu})$$

where the right-side of the implication has to be considered modulo the canonical isomorphism.

PROOF. By Remark 1.2.6 $\Gamma(\rho) \subseteq \Gamma(\pi_{\mathbf{M}^\mu})$, modulo the canonical isomorphism. Thus the statement by Lemma 2.2.5 and (4.3.15). \square

DEFINITION 4.3.20. Let $\mathfrak{V} \doteq \langle \langle \mathfrak{E}, \tau \rangle, \pi, X, \|\cdot\| \rangle$ be a Banach bundle. Let $x_\infty \in X$ and $\mathcal{U}_0 \in \prod_{x \in X_0} \mathcal{C}(\mathbb{R}^+, B_s(\mathfrak{E}_x))$ be such that $\mathcal{U}_0(x)$ is a (C_0) -semigroup of contractions (respectively of isometries) on \mathfrak{E}_x for all $x \in X_0$. Moreover let T_x be the infinitesimal generator of the semigroup $\mathcal{U}_0(x)$ for any $x \in X_0$. Assume $D(T_{x_\infty})$, as defined in Notations 3.1.1, to be dense in \mathfrak{E}_{x_∞} , that $\{v(x) \mid v \in \mathcal{E}\}$ is dense in \mathfrak{E}_x for all $x \in X_0$ and $\exists \lambda_0 > 0$ (respectively $\exists \lambda_0 > 0, \lambda_1 < 0$) such that the range $\mathcal{R}(\lambda_0 - T_{x_\infty})$ is dense in \mathfrak{E}_{x_∞} , (respectively the ranges $\mathcal{R}(\lambda_0 - T_{x_\infty})$ and $\mathcal{R}(\lambda_1 - T_{x_\infty})$ are dense in \mathfrak{E}_{x_∞}). Set

$$(4.3.39) \quad \begin{cases} \mathcal{T}_0(x) \doteq \text{Graph}(T_x), x \in X_0 \\ \Phi \doteq \{\phi \in \Gamma^{x_\infty}(\pi_{\mathbf{E}^\oplus}) \mid (\forall x \in X_0)(\phi(x) \in \mathcal{T}_0(x))\} \\ \mathcal{E} \doteq \{v \in \Gamma(\pi) \mid (\exists \phi \in \Phi)(v(x_\infty) = \phi_1(x_\infty))\} \\ \Theta \doteq \{\prod_{x \in X} \{v(x)\} \mid v \in \mathcal{E}\} \\ T_{x_\infty} : D(T_{x_\infty}) \ni \phi_1(x_\infty) \mapsto \phi_2(x_\infty), \end{cases}$$

where $\langle \langle \mathfrak{E}(\mathbf{E}^\oplus), \tau(\mathbf{E}^\oplus, \mathcal{E}^\oplus) \rangle, \pi_{\mathbf{E}^\oplus}, X, \mathbf{n}^\oplus \rangle$ is the bundle direct sum of the family $\{\mathfrak{V}, \mathfrak{V}\}$. Set

$$\mathcal{U} \in \prod_{x \in X} \mathbf{U}_{\|\cdot\|_{B(\mathfrak{E}_x)}}(\mathcal{L}_{S_x}(\mathfrak{E}_x))$$

(respectively $\mathcal{U} \in \prod_{x \in X} \mathbf{U}_{is}(\mathcal{L}_{S_x}(\mathfrak{E}_x))$) such that $\mathcal{U}(x)$ is the C_0 -semigroup of contractions (respectively of isometries) on \mathfrak{E}_x whose infinitesimal generator is T_x for all $x \in X$. The definitions of T_{x_∞} and $\mathcal{U}(x_\infty)$ are well-posed by Theorem 3.1.16. Finally let \mathcal{T} be defined as in Notations 3.1.1, with \mathcal{T}_0 and Φ given in (4.3.39). Recall that by Theorem 3.1.16

$$\{\langle \mathcal{T}, x_\infty, \Phi \rangle\} \in \text{Graph}(\mathfrak{V}, \mathfrak{V}).$$

THEOREM 4.3.21 (MAIN2). Let X be compact and $\mathfrak{V} \doteq \langle \langle \mathfrak{E}, \tau \rangle, \pi, X, \|\cdot\| \rangle$ be a Banach bundle. Let $x_\infty \in X$ and $\mathcal{U}_0 \in \prod_{x \in X_0} \mathcal{C}(\mathbb{R}^+, B_s(\mathfrak{E}_x))$ be such that $\mathcal{U}_0(x)$ is a (C_0) -semigroup of contractions (respectively of isometries) on \mathfrak{E}_x for all $x \in X_0 \doteq X - \{x_\infty\}$. Moreover let T_x be the infinitesimal generator of the semigroup $\mathcal{U}_0(x)$ for any $x \in X_0$. Assume that $\{v(x) \mid v \in \mathcal{E}\}$ is dense in \mathfrak{E}_x for all $x \in X$ and $\exists \lambda_0 > 0$ (respectively $\exists \lambda_0 > 0, \lambda_1 < 0$) such that the range $\mathcal{R}(\lambda_0 - T_{x_\infty})$ is dense

in \mathfrak{E}_{x_∞} , (respectively the ranges $\mathcal{R}(\lambda_0 - T_{x_\infty})$ and $\mathcal{R}(\lambda_1 - T_{x_\infty})$ are dense in \mathfrak{E}_{x_∞}).
Let us assume the notations of Definition 4.3.20 moreover let $\mathfrak{Z} = \langle \langle \mathfrak{T}, \delta \rangle, \zeta, X, \mathfrak{R} \rangle$ and $\mathfrak{W} \doteq \langle \langle \mathfrak{M}, \gamma \rangle, \rho, X, \mathfrak{R} \rangle$ be such that

(1) \mathfrak{W} is full and $\langle \mathfrak{V}, \mathfrak{W}, X, \mathbb{R}^+ \rangle$ is a (Θ, \mathcal{E}) -structure;

(2) for all $x \in X$

$$\mathcal{C}_{cs}(\mathbb{R}^+, \mathcal{L}_{S_x}(\mathfrak{E}_x)) \subseteq \mathfrak{M}_x \subseteq \mathfrak{L}_1(\mathbb{R}^+, \mathcal{L}_{S_x}(\mathfrak{E}_x); \nu) \bigcap_{\lambda > 0} \mathfrak{L}_1(\mathbb{R}^+, \mathcal{L}_{S_x}(\mathfrak{E}_x); \mu_\lambda);$$

(3) $\langle \mathfrak{V}, \mathfrak{W}(\mathbf{M}^\nu, \Gamma(\rho)), X, Y \rangle$ is invariant and $\langle \mathfrak{V}, \mathfrak{Z}, \mathbf{M}^\nu \rangle$ is ν -related;

(4) $\Gamma(\rho) \star \mathcal{E}(\Theta) \subseteq \Gamma(\zeta)$;

(5) $\mathbf{U}_{\|\cdot\|_{B(\mathfrak{E}_x)}}(\mathcal{L}_{S_x}(\mathfrak{E}_x)) \subseteq \mathfrak{M}_x$ (respectively $\mathbf{U}_{is}(\mathcal{L}_{S_x}(\mathfrak{E}_x)) \subseteq \mathfrak{M}_x$), for all $x \in X$;

(6) $\exists F \in \Gamma(\rho)$ such that $F(x_\infty) = \mathcal{U}(x_\infty)$ and

i: $\langle \mathfrak{V}, \mathfrak{W}, X, \mathbb{R}^+ \rangle$ has the $\mathbf{LD}_{x_\infty}(\{F\}, \mathcal{E})$;

ii: $(\forall v \in \mathcal{E})(\exists \phi \in \Phi)$ s.t. $\phi_1(x_\infty) = v(x_\infty)$ and $(\forall \{z_n\}_{n \in \mathbb{N}} \subset X \mid \lim_{n \in \mathbb{N}} z_n = x_\infty)$ we have that $\{\mathcal{U}(z_n)(\cdot)\phi_1(z_n) - F(z_n)(\cdot)v(z_n)\}_{n \in \mathbb{N}}$ is a bounded equicontinuous sequence;

iii: X is metrizable.

Finally assume that

$$X \ni x \mapsto -T_x \in \text{Cld}(\mathfrak{E}_x),$$

satisfies the property of separation of the spectrum, moreover that there exists a closed curve Γ of which in Definition 4.2.2, for the position $T(x) = -T_x$ for all $x \in X$, and such that

$$\begin{cases} \text{Re}(\Gamma) \subseteq \mathbb{R}^-, \\ -1 \notin \text{Re}(\Gamma). \end{cases}$$

Then

(1) $\mathcal{P} \bullet \Gamma_{\mathcal{E}(\Theta)}^{x_\infty}(\pi) \subseteq \Gamma^{x_\infty}(\pi)$;

(2) If $\mathfrak{D} \doteq \langle \langle \mathfrak{B}, \tau \rangle, \eta, X, \mathfrak{L} \rangle$ is full and $\langle \mathfrak{V}, \mathfrak{D}, X, \{pt\} \rangle$ is an invariant (Θ, \mathcal{E}) -structure such that $\text{Pr}(\mathfrak{E}_x) \subset \mathfrak{B}_x$ for all x then

$$(4.3.40) \quad \mathcal{P} \in \Gamma^{x_\infty}(\eta),$$

and

$$(4.3.41) \quad \boxed{\{\langle \mathcal{T}, x_\infty, \Phi \rangle\} \in \Delta \langle \mathfrak{V}, \mathfrak{D}, \Theta, \mathcal{E} \rangle,}$$

moreover \mathcal{P} satisfies (2.3.3).

Here $\langle \mathfrak{V}, \mathfrak{V}(\mathbf{M}^\nu, \Gamma(\rho)), X, \mathbb{R}^+ \rangle$ is the $(\Theta, \mathcal{E}, \nu)$ – structure underlying $\langle \mathfrak{V}, \mathfrak{W}, X, \mathbb{R}^+ \rangle$. While for all $x \in X$ we set

$$\mathcal{P}(x) \doteq -\frac{1}{2\pi i} \int_{\Gamma} R(-T_x; \zeta) d\zeta \in B(\mathfrak{E}_x),$$

with $R(-T_x; \cdot) : P(-T_x) \ni \zeta \mapsto (-T_x - \zeta)^{-1} \in B(\mathfrak{E}_x)$ the resolvent map of $-T_x$ and $P(-T_x)$ is its resolvent set, while the integration is with respect to the norm topology on $B(\mathfrak{E}_x)$.

PROOF. By the Dupre' Thm., see for example [Kur, Cor. 2.10], and the fact that a metrizable space is completely regular, we deduce by hyp. (6.iii) that \mathfrak{V} is full. So by Prop. 3.1.15 and the density assumption follows that $\text{Dom}(T_{x_\infty})$ is dense in \mathfrak{E}_{x_∞} . Thus we can use the notations in Def. 4.3.20 and by Theorem 3.1.16 we have that $\mathcal{U} \in \Gamma^{x_\infty}(\rho)$, where $\mathcal{U}(x)$ is the C_0 –semigroup generated by T_x , for all $x \in X$. Thus by Proposition 4.3.19 and that $F(x_\infty) = \mathcal{U}(x_\infty)$ (by hyp. (6)) we have

$$(4.3.42) \quad \mathcal{U} \in \Gamma_\diamond^{x_\infty}(\pi_{\mathbf{M}^\nu})$$

Moreover \mathfrak{W} being full implies $\mathfrak{M}_x \subset \mathbf{M}_x^\nu$ so for all $x \in X$

$$(4.3.43) \quad \mathcal{C}_{cs}(\mathbb{R}^+, \mathcal{L}_{S_x}(\mathfrak{E}_x)) \subset \mathbf{M}_x^\nu.$$

Therefore statement (1) follows by Remark 4.3.18 applied to the $(\Theta, \mathcal{E}, \nu)$ – structure underlying $\langle \mathfrak{V}, \mathfrak{W}, X, \mathbb{R}^+ \rangle$, by hyp. (6.i) and by the fact that $F \in \Gamma(\pi_{\mathbf{M}^\nu})$ indeed $\Gamma(\rho) = \Gamma(\pi_{\mathbf{M}^\nu})$ modulo the canonical isomorphism, see Rmk. 1.2.6. Statement (2) follows by (4.3.42), (4.3.25), and $\mathcal{U} = \mathcal{W}_T$ for the position $T(x) \doteq -T_x$ for all $x \in X$. \square

REMARK 4.3.22. By (2.2.6) follows that (4.3.40) is equivalent to $(\exists H \in \Gamma(\eta))(\forall v \in \mathcal{E})$

$$(4.3.44) \quad \lim_{z \rightarrow x_\infty} \|(\mathcal{P}(z) - H(z)) v(z)\| = 0.$$

PROPOSITION 4.3.23. Let E be a Banach space and $S \subseteq \text{Bounded}(E)$ such that $\bigcup_{B \in S} B$ is total in E . Set $\mu_f : \mathcal{C}_{cs}(\mathbb{R}^+) \ni g \mapsto \int_{\mathbb{R}^+} f(s)g(s) ds$ for all $f \in \mathfrak{L}_\infty(\mathbb{R}^+)$. Then

$$(4.3.45) \quad \mathfrak{L}_1(\mathbb{R}^+, B(E)) \subseteq \bigcap_{f \in \mathfrak{L}_\infty(\mathbb{R}^+)} \mathfrak{L}_1(\mathbb{R}^+, \mathcal{L}_S(E), \mu_f).$$

In particular

$$(4.3.46) \quad \mathfrak{L}_1(\mathbb{R}^+, B(E)) \subseteq \bigcap_{\lambda > 0} \mathfrak{L}_1(\mathbb{R}^+, \mathcal{L}_S(E), \eta_\lambda) \bigcap \mathfrak{L}_1(\mathbb{R}^+, \mathcal{L}_S(E), \nu)$$

PROOF. By [INT, Cor.1, N 4, §6, Ch. 4] we deduce that $\mathfrak{L}_\infty(\mathbb{R}^+) \cdot \mathfrak{L}_1(\mathbb{R}^+, B(E)) \subseteq \mathfrak{L}_1(\mathbb{R}^+, B(E))$, hence for all $f \in \mathcal{L}_\infty(\mathbb{R}^+)$.

$$\mathfrak{L}_1(\mathbb{R}^+, B(E)) \subset \mathfrak{L}_1(\mathbb{R}^+, B(E), \mu_f).$$

Moreover by the fact that the norm topology on $B(E)$ is stronger than the lct on $\mathcal{L}_S(E)$ we have for all $\mu \in \text{Radon}(\mathbb{R}^+)$ that

$$\mathfrak{L}_1(\mathbb{R}^+, B(E), \mu) \subseteq \mathfrak{L}_1(\mathbb{R}^+, \mathcal{L}_S(E), \mu).$$

Hence (4.3.45) follows. Finally (4.3.46) follows by (4.3.45). \square

REMARK 4.3.24. Let $\mathfrak{V} \doteq \langle \langle \mathfrak{E}, \tau \rangle, \pi, X, \|\cdot\| \rangle$ be a Banach bundle and $S_x \subseteq \text{Bounded}(\mathfrak{E}_x)$ such that $\bigcup_{B_x \in S_x} B_x$ is total in \mathfrak{E}_x for all $x \in X$. Set

$$(4.3.47) \quad \mathbf{M}_x \doteq \mathfrak{L}_1(\mathbb{R}^+, B(\mathfrak{E}_x)) \cap \mathcal{C}(\mathbb{R}^+, \mathcal{L}_{S_x}(\mathfrak{E}_x)).$$

Thus

$$(4.3.48) \quad \mathbf{U}_{\|\cdot\|_{B(\mathfrak{E}_x)}}(B(\mathfrak{E}_x)) \subset \mathbf{M}_x.$$

Therefore by (4.3.46) in Thm. 4.3.21 we can replace hyp. (2) with (4.3.47), while by (4.3.48) we can replace “ $\mathcal{U}_0 \in \prod_{x \in X_0} \mathcal{C}(\mathbb{R}^+, B_s(\mathfrak{E}_x))$ be such that $\mathcal{U}_0(x)$ is a (C_0) –semigroup of contractions” with “ $\mathcal{U}_0 \in \prod_{x \in X_0} \mathcal{C}(\mathbb{R}^+, B(\mathfrak{E}_x))$ be such that $\mathcal{U}_0(x)$ is a $\|\cdot\|$ –continuous semigroup of contractions” and delete hyp. (4). Similar replacement can be performed in Thm. 3.1.16. Finally notice that in general (4.3.48) do not hold if we replace $\mathbf{U}_{\|\cdot\|_{B(\mathfrak{E}_x)}}(B(\mathfrak{E}_x))$ with $\mathbf{U}_{\|\cdot\|_{B(\mathfrak{E}_x)}}(\mathcal{L}_{S_x}(\mathfrak{E}_x))$, so we cannot take the choice (4.3.47) if we want to have results about C_0 –semigroups.

CHAPTER 5

Constructions

5.1. Kurtz Bundle Construction

In this section we shall construct a special bundle \mathfrak{E} of Banach space such that for it the Main Theorem 3.1.16 reduces to the [Kur, Th. 2.1.] showing in this way that (a particular case) of the construction of Kurtz falls into the general setting of bundle of Ω -spaces.

NOTATIONS 5.1.1. *In this section we shall use the notations of [Kur] with the additional specification of denoting with $\|\cdot\|_n$ the norm in the Banach space L_n . Moreover we denote by X the Alexandrov (one-point) compactification of the locally compact space \mathbb{N} with the discrete topology. Here we recall some definitions given in [Kur]. $\langle L, \|\cdot\| \rangle$ is a Banach space and $\{\langle L_n, \|\cdot\|_n \rangle\}_{n \in \mathbb{N}}$ is a sequence of Banach spaces, moreover $\{P_n \in B(L, L_n)\}_{n \in \mathbb{N}}$ is a sequence of bounded linear operators such that $\forall f \in L$*

$$(5.1.1) \quad \lim_{n \rightarrow \infty} \|P_n f\|_n = \|f\|.$$

Given an element $f \in L$ and a sequence $\{f_n\}_{n \in \mathbb{N}}$ such that $f_n \in L_n$ for all $n \in \mathbb{N}$ we set

$$(5.1.2) \quad \lim_{n \rightarrow \infty} f_n \stackrel{K}{=} f \stackrel{def}{\Leftrightarrow} \lim_{n \rightarrow \infty} \|f_n - P_n f\|_n = 0.$$

In addition to the requirements of [Kur] we assume also that

$$(5.1.3) \quad (\forall n \in \mathbb{N}) (\overline{P_n(L)} = L_n)$$

We shall set here $L_\infty \doteq L$, $\|\cdot\| \doteq \|\cdot\|_\infty$, where $\|\cdot\|$ is the norm on L . Finally for all Z we recall that $B_s(Z)$ is the locally convex space of all linear bounded operators on Z with the strong operator topology.

LEMMA 5.1.2. *Let $f, g \in L$ and $\{f_n\}_{n \in \mathbb{N}}$ such that $f_n \in L_n$ for all $n \in \mathbb{N}$. Then $(\lim_{n \rightarrow \infty} f_n \stackrel{K}{=} f) \wedge (\lim_{n \rightarrow \infty} f_n \stackrel{K}{=} g) \Rightarrow f = g$*

PROOF. Let $(\lim_{n \rightarrow \infty} f_n \stackrel{K}{=} f)$ and $(\lim_{n \rightarrow \infty} f_n \stackrel{K}{=} g)$ thus

$$\lim_{n \in \mathbb{N}} \|P_n(f - g)\| \leq \lim_{n \in \mathbb{N}} \|P_n f - f_n\| + \lim_{n \in \mathbb{N}} \|P_n g - f_n\| = 0,$$

so the statement follows by (5.1.1). \square

DEFINITION 5.1.3. *Set*

$$\begin{cases} \mathbf{L} \doteq \{\langle L_x, \|\cdot\|_x \rangle\}_{x \in X}, \\ \mathcal{E}(L) \doteq \{\sigma^f \mid f \in L\}, \end{cases}$$

where $\sigma^f \in \prod_{x \in X} L_x$ such that $\sigma^f(n) \doteq P_n f$ for all $n \in \mathbb{N}$ and $\sigma^f(\infty) \doteq f$.

DEFINITION 5.1.4. *By (5.1.1) the sequence $\{\|P_n f\|_n\}_{n \in \mathbb{N}}$ is bounded for all $f \in L$ so $\sigma^f \in \prod_{x \in X}^b L_x$. Moreover by (5.1.1) $\mathcal{E}(L)$ satisfies $FM(4)$, finally by the request (5.1.3) it satisfies also $FM(3)$. Therefore we can define the Kurtz bundle the following bundle*

$$\mathfrak{V}(\mathbf{L}, \mathcal{E}(L))$$

generated by the couple $\langle \mathbf{L}, \mathcal{E}(L) \rangle$, see in Def. 1.2.5.

REMARK 5.1.5. *By Remark 1.2.6 and the compactness of X by construction, we have that*

$$(5.1.4) \quad \mathcal{E}(L) \simeq \Gamma(\pi_{\mathbf{L}}).$$

Finally by applying the principle of uniform boundedness, [Kat, Th. 1.29, No3, Ch.3], we deduce that the sequence $\{\|P_n\|_{B(L, L_n)}\}_{n \in \mathbb{N}}$ is bounded.

DEFINITION 5.1.6. *Fix $\mathcal{U}_0 \in \prod_{n \in \mathbb{N}} \mathcal{C}(\mathbb{R}^+, B_s(L_n))$ such that $\mathcal{U}_0(x)$ is a (C_0) -semigroup of isometries on L_n for all $n \in \mathbb{N}$. Denote by T_n the infinitesimal generator of the semigroup $\mathcal{U}_0(n)$ for any $n \in \mathbb{N}$. Let us take the positions (3.1.18), where $\langle \langle \mathfrak{E}(\mathbf{E}^\oplus), \tau(\mathbf{E}^\oplus, \mathcal{E}^\oplus) \rangle, \pi_{\mathbf{E}^\oplus}, X, \mathfrak{n}^\oplus \rangle$ is the bundle direct sum of the family $\{\mathfrak{V}(\mathbf{L}, \mathcal{E}(L)), \mathfrak{V}(\mathbf{L}, \mathcal{E}(L))\}$. In addition we maintain the Notations 3.1.1 where \mathfrak{V} has to be considered the Kurtz bundle and $x_\infty \doteq \infty$, thus $\mathcal{T} \in \prod_{x \in X} \text{Graph}(L_x \times L_x)$ so that $\mathcal{T} \upharpoonright X - \{\infty\} \doteq \mathcal{T}_0$ and*

$$\mathcal{T}(\infty) \doteq \{\phi(\infty) \mid \phi \in \Phi\},$$

and $D(T_\infty) \doteq \text{Pr}_1^\infty(\mathcal{T}(\infty)) = \{\phi_1(\infty) \mid \phi \in \Phi\}$. Finally $\mathcal{S} \doteq \{S_x\}_{x \in X}$ where $(\forall B \in \Theta)(\forall x \in X)$

$$(5.1.5) \quad \begin{cases} \mathbf{D}(B, \mathcal{E}) \doteq \mathcal{E} \cap (\prod_{x \in X} B_x) \\ \mathcal{B}_B^x \doteq \{v(x) \mid v \in \mathbf{D}(B, \mathcal{E})\} \\ S_x \doteq \{\mathcal{B}_B^x \mid B \in \Theta\}. \end{cases}$$

PROPOSITION 5.1.7. Let $\bar{f} \in \prod_{x \in X} L_x$ Thus

$$\lim_{n \rightarrow \infty} \bar{f}(n) \stackrel{K}{=} \bar{f}(\infty) \Leftrightarrow \bar{f} \in \Gamma^\infty(\pi_{\mathbf{L}}).$$

PROOF. By (5.1.4) and implication (3) \Rightarrow (1) of Corollary 1.2.10 we have that $\lim_{n \rightarrow \infty} \bar{f}(n) \stackrel{K}{=} \bar{f}(\infty)$ implies that

$$\bar{f} \text{ is continuous at } \infty,$$

indeed $\sigma^{\bar{f}(\infty)} \in \Gamma(\pi_{\mathbf{L}})$ modulo isomorphism. By the upper semicontinuity of $\|\cdot\| : \mathfrak{E} \rightarrow \mathbb{R}^+$, due to the construction of the bundle $\mathfrak{V}(\mathbf{L}, \mathcal{E}(L))$ and to [Kur, 1.6.(ii)], and by the fact that the composition of any u.s.c. map with any continuous one at a point is an u.s.c. map in the same point, we deduce that $\|\cdot\| \circ \bar{f}$ is u.s.c. at ∞ . Thus $\sup_{x \in X} \|\bar{f}(x)\|_x < \infty$, indeed we applied to the u.s.c. map $\|\cdot\| \circ \bar{f}$ the fact that X is compact (so quasi compact), $-\|\cdot\| \circ \bar{f}$ is l.s.c, the [GT, Th. 3, §6.2., Ch. 4] and [GT, form.(2), §5.4., Ch. 4]. Therefore

$$\bar{f} \in \prod_{x \in X}^b L_x.$$

Then $\bar{f} \in \Gamma^\infty(\pi_{\mathbf{L}})$. The remaining implication follows by Corollary 1.2.10 and by the fact that $\mathfrak{V}(\mathbf{L}, \mathcal{E}(L))$ is full, X being locally compact and by the fact that any Banach bundle over a locally compact space is full, see [FD, Appendix C]. \square

PROPOSITION 5.1.8. We have

$$\Gamma^\infty(\pi_{\mathbf{L}^\oplus}) = \left\{ \sigma_1 \oplus \sigma_2 \mid \sigma_i \in \prod_{x \in X} L_x, \lim_{n \rightarrow \infty} \sigma_i(n) \stackrel{K}{=} \sigma(\infty), i = 1, 2 \right\}.$$

Here, we used the Convention 1.2.28 and set $(\sigma_1 \oplus \sigma_2)(x) \doteq \sigma_1(x) \oplus \sigma_2(x)$.

PROOF. By Convention 1.2.28 and Corollary 1.2.27 $\sigma_1 \oplus \sigma_2$ is continuous at ∞ if and only if σ_i is continuous at ∞ for all $i = 1, 2$. Thus the statement by Proposition 5.1.7. \square

PROPOSITION 5.1.9. Let $\mathcal{U}_0 \in \prod_{n \in \mathbb{N}} \mathcal{C}(\mathbb{R}^+, B_s(L_n))$ be such that $\mathcal{U}_0(x)$ is a (C_0) -semigroup of contractions on L_n for all $n \in \mathbb{N}$. Moreover let us denote by T_n the infinitesimal generator of the semigroup $\mathcal{U}_0(n)$ for any $n \in \mathbb{N}$. Thus with the positions (3.1.18) where \mathfrak{V} is the Kurtz bundle we have

$$(5.1.6) \quad \begin{cases} \Phi = \{\sigma_1 \oplus \sigma_2 \mid (\forall i \in \{1, 2\})(\sigma_i \in \prod_{x \in X} L_x)(1 - 2)\} \\ (1) \lim_{n \rightarrow \infty} \sigma_i(n) \stackrel{K}{=} \sigma_i(\infty) \\ (2) (\forall n \in \mathbb{N})(\sigma_1(n), \sigma_2(n)) \in \text{Graph}(T_n), \end{cases}$$

and

$$(5.1.7) \quad \begin{cases} \mathcal{E} = \{\sigma^{\sigma_1(\infty)} \mid \sigma_1 \in \prod_{x \in X} L_x(1 - 2 - 3)\} \\ (1) \lim_{n \rightarrow \infty} \sigma_1(n) \stackrel{K}{=} \sigma_1(\infty) \\ (2) (\forall n \in \mathbb{N})(\sigma_1(n) \in \text{Dom}(T_n)) \\ (3) (\exists f \in L_\infty)(\lim_{n \rightarrow \infty} T_n \sigma_1(n) \stackrel{K}{=} f). \end{cases}$$

Moreover $\exists! f$ satisfying (3) in (5.1.7) and $(\forall \sigma_1 \in \mathcal{E})(\sigma_1, \sigma_2) \in \Phi$, where $\sigma_2 \in \prod_{x \in X} L_x$ such that $(\forall n \in \mathbb{N})(\sigma_2(n) \doteq T_n \sigma_1(n))$ and $\sigma_2(\infty) \doteq f$.

PROOF. The first sentence follows by Proposition 5.1.8, while the second comes by the first one and Lemma 5.1.2. \square

ASSUMPTIONS 5.1.10. We assume $\exists \{I_n \in B(L_n, L)\}_{n \in \mathbb{N}}$ such that

$$(5.1.8) \quad \begin{cases} \sup_{n \in \mathbb{N}} \|I_n\|_{B(L_n, L)} < \infty, \\ (\forall f \in L)(\forall n \in \mathbb{N})(I_n \circ P_n = Id). \end{cases}$$

Moreover we assume that

$$(5.1.9) \quad \overline{\lim}_{n \rightarrow \infty} \|P_n\| \leq 1.$$

In addition we assume that $(\forall g \in L)(\exists \sigma_1 \in \prod_{x \in X} L_x)$ such that

$$(5.1.10) \quad \begin{cases} (1) \lim_{n \rightarrow \infty} \sigma_1(n) \stackrel{K}{=} \sigma_1(\infty) \\ (2) (\forall n \in \mathbb{N})(\sigma_1(n) \in \text{Dom}(T_n)) \\ (3) (\exists f \in L_\infty)(\lim_{n \rightarrow \infty} T_n \sigma_1(n) \stackrel{K}{=} f) \\ (4) g = \sigma_1(\infty). \end{cases}$$

Set

$$(5.1.11) \quad \mathfrak{U} \doteq \{F \in \mathcal{C}(\mathbb{R}^+, B_s(L)) \mid (\forall s \in \mathbb{R}^+)(\forall v \in L)(\|F(s)v\| = \|v\|)\}.$$

In the following definition we shall give the data for constructing a bundle \mathfrak{W} such that $\langle \mathfrak{V}(\mathbf{L}, \mathcal{E}(L)), \mathfrak{W}, X, \mathbb{R}^+ \rangle$ would be a (Θ, \mathcal{E}) –structure.

DEFINITION 5.1.11. Set $P_\infty \doteq I_\infty \doteq Id : L \rightarrow L$, moreover $\forall U \in \mathfrak{U}$ set $F_U \in \prod_{x \in X} \mathcal{C}_c(\mathbb{R}^+, \mathcal{L}_{S_x}(L_x))$ such that $\forall x \in X$

$$\begin{cases} F_U(x) \doteq P_x \circ U(\cdot) \circ I_x, \\ P_x \circ U(\cdot) \circ I_x : \mathbb{R}^+ \ni s \mapsto P_x \circ U(s) \circ I_x \in B(L_x). \end{cases}$$

Now we can define $\forall x \in X$

$$\mathbf{M}_x \doteq \text{span} \{F_U(x) \mid U \in \mathfrak{U}\}.$$

\mathbf{M}_x has to be considered as Hlcs with the topology induced by that on $\mathcal{C}_c(\mathbb{R}^+, \mathcal{L}_{S_x}(L_x))$.

¹ Moreover set

$$\mathcal{M} \doteq \text{span} \{F_U \mid U \in \mathfrak{U}\}.$$

THEOREM 5.1.12. \mathbf{M}_x as Hlcs is well-defined for any $x \in X$, moreover $\mathcal{M} \subset \prod_{x \in X}^b \mathbf{M}_x$ and $\mathbf{M}_x = \{F(x) \mid F \in \mathcal{M}\}$. Finally \mathcal{M} satisfies FM(3) – FM(4) with respect to \mathbf{M} .

PROOF. By Remark 3.1.12 we have that $\mathcal{C}_c(\mathbb{R}^+, B_s(L_x)) \subset \mathcal{C}_c(\mathbb{R}^+, \mathcal{L}_{S_x}(L_x))$ hence for the first sentence of the statement it is sufficient to show that $P_x \circ U(\cdot) \circ I_x \in \mathcal{C}_c(\mathbb{R}^+, B_s(L_x))$ for any $U \in \mathfrak{U}$. For $x = \infty$ is trivial so let $n \in \mathbb{N}$ and $f_n \in L_n$ thus for all $s \in \mathbb{R}^+$ and all net $\{s_\alpha\}_{\alpha \in D}$ in \mathbb{R}^+ converging at s we have

$$\lim_{\alpha \in D} \|P_n \circ U(s_\alpha) \circ I_n(f_n) - P_n \circ U(s) \circ I_n(f_n)\|_n = \lim_{\alpha \in D} \|P_n(U(s_\alpha) - U(s))I_n f_n\|_n = 0,$$

where we used the fact that U is strongly continuous and P_n is norm continuous by construction. Thus the first sentence of the statement follows. Let $v \in \mathcal{E}$ and $U \in \mathfrak{U}$

¹ $\mathcal{C}_c(\mathbb{R}^+, \mathcal{L}_{S_x}(L_x))$ is Hausdorff for all $x \in X$ by the fact that $\bigcup_{B \in \Theta} \mathcal{B}_B^x = L_x$, see later Prop. 5.1.15.

thus $\forall K \in \text{Compact}(\mathbb{R}^+)$

$$\begin{aligned} \sup_{n \in \mathbb{N}} \sup_{s \in K} \|P_n U(s) I_n v(n)\|_n &\leq M \sup_{n \in \mathbb{N}} \sup_{s \in K} \|U(s) I_n v(n)\|_\infty \\ &= M \sup_{n \in \mathbb{N}} \|I_n v(n)\|_\infty \\ &\leq M \sup_{n \in \mathbb{N}} \|I_n\| \sup_{n \in \mathbb{N}} \|v(n)\|_\infty < \infty. \end{aligned}$$

Here $M \doteq \sup_{n \in \mathbb{N}} \|P_n\|$, in the second one the hypothesis that $U(s)$ is an isometry for all $s \in \mathbb{R}^+$, in the final inequality we considered (5.1.8), $\mathcal{E} \subset \prod_{x \in X}^b L_x$ and that $M < \infty$ by Remark 5.1.5. Therefore by Remark 3.1.12 $\mathcal{M} \subset \prod_{x \in X}^b \mathbf{M}_x$. The equality $\mathbf{M}_x = \{F(x) \mid F \in \mathcal{M}\}$ comes by construction, in particular \mathcal{M} satisfies the $FM(3)$ with respect to the \mathbf{M} . $\forall v \in \mathcal{E}$

$$\begin{aligned} \overline{\lim}_{n \rightarrow \infty} \sup_{s \in K} \|P_n U(s) I_n v(n)\|_n &\leq \overline{\lim}_{n \rightarrow \infty} \left(\|P_n\| \sup_{s \in K} \|U(s) I_n v(n)\|_n \right), [\mathbf{GT}, \text{Prop. 11, §5.6. Ch. 4}] \\ &\leq \overline{\lim}_{n \rightarrow \infty} \|P_n\| \overline{\lim}_{n \rightarrow \infty} \sup_{s \in K} \|U(s) I_n v(n)\|_n, [\mathbf{GT}, \text{Prop. 13, §5.7. Ch. 4}] \\ &\leq \overline{\lim}_{n \rightarrow \infty} \|I_n v(n)\|_\infty, \quad (5.1.9), (5.1.11) \\ &= \overline{\lim}_{n \rightarrow \infty} \|I_n P_n f\|_\infty, \quad v \in \mathcal{E} \subset \Gamma(\pi) \simeq \mathcal{E}(L) \\ &= \|f\|_\infty, \quad (5.1.8) \\ &= \|v(\infty)\|_\infty. \end{aligned}$$

Thus by considering that U is a map of isometries we have

$$\overline{\lim}_{n \rightarrow \infty} \sup_{s \in K} \|P_n U(s) I_n v(n)\|_n \leq \sup_{s \in K} \|P_\infty U(s) I_\infty v(\infty)\|_\infty.$$

Hence by $[\mathbf{GT}, \text{Prop. 3, §7.1. Ch. 4}]$ and $[\mathbf{GT}, (13), §5.6. Ch. 4]$ we deduce that

$$X \ni x \mapsto \sup_{s \in K} \|P_x U(s) I_x v(x)\|_x \text{ is } u.s.c. \text{ at } \infty,$$

therefore it is *u.s.c.* on X because of it is continuous in each point in \mathbb{N} due to the fact that the topology induced on \mathbb{N} by that on X is the discrete topology. So \mathcal{M} satisfies the $FM(4)$ with respect to the \mathbf{M} . \square

DEFINITION 5.1.13. *Theorem 5.1.12 allows us to construct a bundle of Ω –space namely the bundle $\mathfrak{V}(\mathbf{M}, \mathcal{M})$ generated by the couple $\langle \mathbf{M}, \mathcal{M} \rangle$, see Def 1.2.5.*

REMARK 5.1.14. *By Remark 1.2.6 and the compactness of X we have*

$$(5.1.12) \quad \mathcal{M} \simeq \Gamma(\pi_{\mathbf{M}}).$$

Hence by $\mathbf{M}_x = \{F(x) \mid F \in \mathcal{M}\}$ we have that $\mathfrak{V}(\mathbf{M}, \mathcal{M})$ is full.

PROPOSITION 5.1.15. We have that $\overline{\bigcup_{B \in \Theta} \mathcal{B}_B^x} = L_x$ for all $x \in X$ moreover $\langle \mathfrak{V}(\mathbf{L}, \mathcal{E}(L)), \mathfrak{V}(\mathbf{M}, \mathcal{M}), X, \mathbb{R}^+ \rangle$ is a (Θ, \mathcal{E}) -structure.

PROOF. By assumptions (5.1.10), (5.1.3), Proposition 5.1.9 and Remark 3.1.12 we obtain that $\overline{\bigcup_{B \in \Theta} \mathcal{B}_B^x} = L_x$ for all $x \in X$. The remaining requests for the second sentence of the statement come by the construction of \mathcal{M} and \mathbf{M} . \square

COROLLARY 5.1.16. If $D(T_{x_\infty})$ is dense in \mathfrak{E}_{x_∞} , and $\exists \lambda_0 > 0, \lambda_1 < 0$ such that the ranges $\mathcal{R}(\lambda_0 - T_{x_\infty})$ and $\mathcal{R}(\lambda_1 - T_{x_\infty})$ are dense in \mathfrak{E}_{x_∞} , then $\langle T, \infty, \Phi \rangle \in \text{Graph}(\mathfrak{V}(\mathbf{L}, \mathcal{E}(L)), \mathfrak{V}(\mathbf{L}, \mathcal{E}(L)))$ and the following

$$T_\infty : D(T_\infty) \ni \phi_1(\infty) \mapsto \phi_2(\infty)$$

is a well-defined operator which is the generator of a C_0 -semigroup of isometries on \mathfrak{E}_∞ .

PROOF. By Propositions 5.1.15 and 5.1.19 we have that the first part of hypotheses of Theorem 3.1.16 is satisfied so the statement by the first sentence of the statement of Theorem 3.1.16. \square

DEFINITION 5.1.17. Let us denote by \mathcal{U}_∞ the C_0 -semigroup of isometries on L_∞ . Moreover set $\mathcal{U} \in \prod_{x \in X} \mathbf{U}_{is}(B_s(L_x))$ such that $\mathcal{U} \upharpoonright \mathbb{N} = \mathcal{U}_0$ and $\mathcal{U}(\infty) = \mathcal{U}_\infty$.

THEOREM 5.1.18. $(\exists F \in \Gamma(\pi_{\mathbf{M}}))(F(\infty) = \mathcal{U}(\infty))$ such that $(\forall v \in \mathcal{E})(\exists \phi \in \Phi)$ s.t. $\phi_1(x_\infty) = v(x_\infty)$ and $(\forall \{z_n\}_{n \in \mathbb{N}} \subset X \mid \lim_{n \in \mathbb{N}} z_n = x_\infty)$ we have that $\{\mathcal{U}(z_n)(\cdot)\phi_1(z_n) - F(z_n)(\cdot)v(z_n)\}_{n \in \mathbb{N}}$ is a bounded equicontinuous sequence. Moreover we can choose F such that $F = F_{\mathcal{U}_\infty}$.

PROOF. By Prop. 5.1.9 and (5.1.12) the statement is equivalent to show that $\forall \sigma_1 \in \prod_{x \in X} L_x$ satisfying (1 – 2 – 3) of (5.1.7) and $(\forall \{z_n\}_{n \in \mathbb{N}} \subset X \mid \lim_{n \in \mathbb{N}} z_n = \infty)$ we have that

$$(5.1.13) \quad \{\mathcal{U}(z_n)(\cdot)\sigma_1(z_n) - F_{\mathcal{U}_\infty}(z_n)(\cdot)\sigma^{\sigma_1(\infty)}(z_n)\}_{n \in \mathbb{N}}$$

is a bounded equicontinuous sequence. Moreover by the second assumption (5.1.8) and (5.1.13)

$$(5.1.14) \quad \{\mathcal{U}(z_n)(\cdot)\sigma_1(z_n) - P_{z_n}\mathcal{U}_\infty(z_n)(\cdot)\sigma_1(\infty)\}_{n \in \mathbb{N}}$$

is a bounded equicontinuous sequence. Set $\sigma_2 \in \prod_{x \in X} L_x$ such that $\sigma_2(x) \doteq T_x \sigma_1(x)$, for all $x \in X$, thus

$$\sigma_i \in \Gamma^\infty(\pi_{\mathbf{L}}),$$

for all $i = 1, 2$, indeed for $i = 1$ follows by (1) of (5.1.7) and Prop. 5.1.7, while for $i = 2$ follows by construction of T_∞ , the second sentence of Prop. 5.1.9, the fact that by construction $\Phi \subseteq \Gamma(\pi_{\mathbf{E}^\oplus})$, see (3.1.18), and finally by Corollary 1.2.27. Therefore in particular σ_i is continuous at ∞ . Thus by considering that $\sigma^{\sigma_i(\infty)} \in \Gamma(\pi_{\mathbf{L}})$ modulo isomorphism by (5.1.4), that $\mathfrak{V}(\mathbf{L}, \mathcal{E}(L))$ is full being a Banach bundle over a locally compact space, we deduce by Prop. 1.2.8

$$\lim_{n \in \mathbb{N}} \|\sigma_i(z_n) - \sigma^{\sigma_i(\infty)}(\pi \circ \sigma_i(z_n))\|_{\pi \circ \sigma_i(z_n)} = 0.$$

Then by considering that $\pi \circ \sigma_i = Id$ because of σ_i is a selection, we have

$$(5.1.15) \quad \lim_{n \in \mathbb{N}} \|\sigma_i(z_n) - P_{z_n} \sigma_i(\infty)\|_{z_n} = 0.$$

The statement now follows by (5.1.15), (5.1.14) and by using the same argumentation used in proof of [Kur, Th. 1.2] for proving a similar result. \square

PROPOSITION 5.1.19. *With the notations of Def. 3.1.8 we have that*

$$\mathbf{M}_x \subset \bigcap_{\lambda > 0} \mathfrak{L}_1(\mathbb{R}^+, \mathcal{L}_{S_x}(L_x); \mu_\lambda),$$

and (3.1.14) holds.

PROOF. By Proposition 3.3.2. \square

THEOREM 5.1.20. *$\langle \mathfrak{V}(\mathbf{L}, \mathcal{E}(L)), \mathfrak{V}(\mathbf{M}, \mathcal{M}), X, \mathbb{R}^+ \rangle$ has the full Laplace duality property, moreover $\forall U \in \mathbf{U}_{1, \|\cdot\|}(B_s(L)), \forall \lambda > 0$ and $\forall f \in L$ we have that*

$$\mathfrak{L}(F_U)(\cdot)(\lambda) \sigma^f(\cdot) = \sigma^{(\lambda - T_U)^{-1} f}.$$

Here T_U is the infinitesimal generator of the semigroup U .

PROOF. Let $f \in L$ and $U \in \mathfrak{U}$ thus for all $x \in X$ and $\lambda > 0$ we have

$$(5.1.16) \quad \begin{aligned} \int_0^\infty e^{-\lambda s} P_x U(s) I_x \sigma^f(x) ds &= \int_0^\infty e^{-\lambda s} P_x U(s) f ds \\ &= P_x \int_0^\infty e^{-\lambda s} U(s) f ds, \end{aligned}$$

where the first equality follows by the second assumption 5.1.8, while the second one by the linearity and continuity of P_x and by [INT, Prop.1, No1, §1, Ch. 6]. Thus the first

sentence of the statement by (5.1.4) and (5.1.12). The second sentence of the statement follows by the (5.1.16) and by Hille-Yosida Theorem, see [Kur, Th. 1.2.]. \square

COROLLARY 5.1.21. *Let us assume the hypotheses of Corollary 5.1.16. Then $(\forall g \in L)(\forall K \in Compact(\mathbb{R}^+))$*

$$(5.1.17) \quad \lim_{z \rightarrow \infty} \sup_{s \in K} \|(\mathcal{U}(z)(s) \circ P_z - P_z \circ \mathcal{U}_\infty(s))g\| = 0.$$

Moreover

$$(5.1.18) \quad \mathcal{U} \in \Gamma^\infty(\rho).$$

In particular

$$(5.1.19) \quad \{\langle \mathcal{T}, \infty, \Phi \rangle\} \in \Delta_\Theta \langle \mathfrak{V}(\mathbf{L}, \mathcal{E}), \mathfrak{V}(\mathbf{M}, \mathcal{M}), \mathcal{E}, X, \mathbb{R}^+ \rangle.$$

PROOF. By Proposition 5.1.19 follows (3.1.14), hypothesis (i) of Theorem 3.1.16 follows by Theorem 5.1.20, (ii) by Theorem (5.1.18), finally (iii) follows by [GT, Coroll. of Prop.16, §2.9, Ch. 9] and by the fact that $\{\{n\} \mid n \in \mathbb{N}\}$ is a base for the topology on \mathbb{N} . Thus by Theorem 3.1.16 we obtain (5.1.18), (5.1.19) and $(\forall v \in \mathcal{E})(\forall K \in Compact(\mathbb{R}^+))$

$$(5.1.20) \quad \lim_{z \rightarrow \infty} \sup_{s \in K} \|\mathcal{U}(z)(s)v(z) - F(z)(s)v(z)\| = 0,$$

where F is any map of which in Theorem 5.1.18. Now by Theorem 5.1.18 we can take in the previous equation $F = F_{\mathcal{U}_\infty}$, moreover by (5.1.7) and assumption (5.1.10) we have

$$\mathcal{E} = \{\sigma^g \mid g \in L\},$$

therefore by (5.1.8) $\forall s \in \mathbb{R}^+, \forall z \in X$ and $\forall g \in L$

$$F_{\mathcal{U}_\infty}(z)\sigma^g(z) = (P_z \circ \mathcal{U}_\infty(s))g.$$

Hence by (5.1.20) follows (5.1.17). \square

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